Supplementary Materials for "Estimation and Inference for Spot Variance of Microstructure Noise"

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Abstract

This note contains supplementary materials for Li and Zhang (2026). Section SA proves the theoretical results of Li and Zhang (2026), together with the auxiliary results and technical lemmas they require. Section SB documents the trade filter, the announcement groups, and the empirical implementation, and examines the sensitivity of announcement-time and midday rejection frequencies to fixed horizons and spot-window caps. Section SC reports additional simulation results under other noise amplitudes and degrees of serial dependence. References to equations, theorems, lemmas, tables, and figures prefixed with SA, SB, or SC (e.g., Lemma SA.3, Table SB.1, Figure SC.1) pertain to objects within this supplement; unprefixed numbering denotes corresponding objects in the main paper.

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SA Mathematical Proofs

This section proves Theorems 3.1 to 3.3 and Corollary 3.1 of Li and Zhang (2026), together with Theorem B.1. Section SA.1 collects the setting and the technical conditions, Section SA.2 states the auxiliary results used directly in the proofs, and Section SA.3 contains the proofs of the main results. Section SA.4 proves the auxiliary results and the further lemmas they require, and Section SA.5 proves Theorem B.1.

SA.1 Setting and technical conditions

In the following proofs, we follow Jacod, Li, and Zheng (2017) and Li and Linton (2022) to let $\Omega = \Omega^{(0)} \times \Omega^{(1)}$, $\mathcal{F} = \mathcal{F}^{(0)} \otimes \mathcal{G}$ and $\mathbb{P} = \mathbb{P}^{(0)} \otimes \mathbb{P}^{(1)}$, where $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^{(0)})$ is the filtered probability space where X, α, Γ and the observation times are defined, with $\mathcal{F}^{(0)} = \mathcal{F}_\infty \vee \sigma(t_i^n : i \geq 0,\ n \geq 1)$ as in Assumption (N). The observation times are stopping times for the enlarged filtrations $(\mathcal{F}_t^n)_{t\geq 0}$ defined in Assumption (O). Let K denote a constant independent of n, which can change across lines or within one line. Throughout, $\xrightarrow{\mathcal{L}_s}$ denotes $\mathcal{F}_\infty$-stable convergence in law, where $\mathcal{F}_\infty = \bigvee_{s\geq 0} \mathcal{F}_s$.

We write $\Gamma_i^n := \Gamma_{t_i^n}$, so that $(\varepsilon_i^n)^2 = \Gamma_i^n \chi_i^2$ by (5). For $t \in \mathcal{T}$ and $\varsigma \in \{+, -\}$, we write $\Gamma_{t,\varsigma}$, $\alpha_{t,\varsigma}$, $\widetilde{\sigma}_{t,\varsigma}$, $\widehat{\mathbb{I}}_{t,\varsigma}^n$, $\mathbb{I}_{t,\varsigma}^n$, $\widetilde{\Gamma}_{t,\varsigma}^n$, $\widehat{\Gamma}_{t,\varsigma}^n$, $\mathfrak{S}_{t,\varsigma}$ and $\widehat{\mathfrak{S}}_{t,\varsigma}^n$ for the corresponding quantities in Section 3 and Appendix B of Li and Zhang (2026) with subscript $t+$ if $\varsigma = +$ and $t-$ if $\varsigma = -$.

For any processes V and U and $k \in \mathbb{N}_+$, denote $f(V, U; k)_i^n := \left(V_i^n - V_{i-k}^n\right)\left(U_i^n - U_{i+k}^n\right)$ and $\overline{f}(V, U)_i^n := \frac{1}{p_n}\sum_{p=0}^{p_n-1} f(V, U; k_{n,p})_i^n$, where $k_{n,p} = k_n + p$. When $V = U$, we further simplify the notations to $f(V; k)_i^n$ and $\overline{f}(V)_i^n$, respectively. Note that $f(V, U; k)_i^n$ reduces to $f(k)_i^n$ defined in Section 3.2 for $V = U = Y$.

Assumption (K). *Let Z satisfy Assumption (H). The process Z and its component processes b^Z, σ^Z, and δ^Z are bounded with $|\delta^Z(\omega, t, z)| \leq J^Z(z)$ for some bounded function $J^Z(z)$ satisfying $\int_{\mathbb{R}}(J^Z(z))^2\lambda(\mathrm{d}z) < \infty$.*

By standard localization, including localization of the full variance process away from zero as permitted by (4), it suffices to work under the following bounded assumptions.

Assumption (S-HONV). *Let Assumption (O) and Assumption (N) hold. We assume that Assumption (K) holds for X, α, and Γ', that the process $1/\alpha$ is bounded, and that the bounds in (3) hold for all $i \geq 1$ without the restriction $\{t_{i-1}^n < \tau_m\}$, with constants not depending on m. Furthermore, we assume that Assumption (V) holds, with bounded scheduled jump sizes φ_ℓ. There are deterministic constants $0 < \underline{\Gamma} < \overline{\Gamma} < \infty$ such that $\underline{\Gamma} \leq \Gamma_s \leq \overline{\Gamma}$ for $0 \leq s \leq 1$, almost surely.*

Throughout the proofs, we write $\widetilde{\sigma} := \sigma^{\Gamma'}$ and $W := W^{\Gamma'}$ for the volatility process and the driving Brownian motion of Γ' in its representation (2). Under Assumption (N), the discrete part of the noise $\{\chi_i\}_{i\in\mathbb{Z}}$ is defined on $(\Omega^{(1)}, \mathcal{G}, \mathbb{P}^{(1)})$ and is independent of the continuous-time processes. Write $\mathcal{G}_m := \sigma(\chi_i : i \leq m)$ and $\mathcal{G}^m := \sigma(\chi_i : i \geq m)$, and set $\rho_0 = 1$. For any $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$, $\xi \in$

$\mathbb{L}_2(\mathcal{G}_m)$, and $\xi' \in \mathbb{L}_2(\mathcal{G}^{m+\ell})$ (see, e.g., (VIII.3.101) and Lemma VIII.3.102(c) of Jacod and Shiryaev (2003)),

$$|\mathbf{Cov}(\xi, \xi')| \leq \rho_\ell \|\xi\|_2 \|\xi'\|_2, \quad \|\mathbb{E}(\xi'|\mathcal{G}_m) - \mathbb{E}(\xi')\|_2 \leq \rho_\ell \|\xi'\|_2. \tag{SA.1}$$

SA.2 Auxiliary results

We now state the auxiliary results used directly in the proofs of the main results; their proofs are given in Section SA.4. The first lemma controls the distance between a fixed time and the neighboring observation times.

Lemma SA.1. *Let Assumption (S-HONV) hold. There exist deterministic constants K and n_0, independent of s, ℓ, x, such that, for all $n \geq n_0$, $s \in (0,1)$, $x > 0$ and $\ell \in \mathbb{N}_+$,*

$$\mathbb{E}\big(t^n_{n_s+\ell} - s\big) \leq K\ell\Delta_n, \qquad \mathbb{E}\big(s - t^n_{n_s}\big) \leq K\Delta_n, \tag{SA.2}$$

$$\mathbb{P}(t^n_{n_s+\ell} - s \geq x) \leq \frac{K\ell\Delta_n}{x}. \tag{SA.3}$$

Next, we introduce a construction used to control the pre-t windows. Under Assumption (S-HONV), there exist $0 < \underline{\alpha} < \overline{\alpha} < \infty$ such that $\underline{\alpha} \leq \alpha^n_{i-1} \leq \overline{\alpha}$ for $t^n_{i-1} \in (0,1)$. For an arbitrary sequence of deterministic integers $\{j_n\}_n$, we let $h_n := Cj_n\Delta_n$, where $C > 0$ is a fixed constant such that $C\underline{\alpha} > 2$. For $t \in \mathcal{T}$, let $a_n := n_{t-h_n} + 1$, $J^n_t := n_t - a_n$ and $\widetilde{J}^n_t := \lfloor \alpha_{t-h_n} h_n/\Delta_n \rfloor - 1 = \lfloor C\alpha_{t-h_n} j_n \rfloor - 1$.

Lemma SA.2. *Under Assumption (S-HONV), if $j_n \to \infty$ and $j_n\Delta_n \to 0$, then, for every $t \in \mathcal{T}$,*

$$\frac{J^n_t - \widetilde{J}^n_t}{j_n} \xrightarrow{\mathbb{P}} 0. \tag{SA.4}$$

Moreover, there exist deterministic integers $b_n \to \infty$, common to all $t \in \mathcal{T}$, with $b_n/j_n \to 0$, such that, for the event $\mathcal{A}^n_t := \{|J^n_t - \widetilde{J}^n_t| \leq b_n\}$,

$$\mathbb{P}\left(\bigcap_{t\in\mathcal{T}} \mathcal{A}^n_t\right) \to 1. \tag{SA.5}$$

Next, we decompose the error of the oracle estimator. For $(t,\varsigma) \in \mathcal{T} \times \{+,-\}$, we have $\widetilde{\Gamma}^n_{t,\varsigma} - \Gamma_{t,\varsigma} = G^n_{t,\varsigma} + H^n_{t,\varsigma} + D^n_{t,\varsigma}$, where

$$G^n_{t,\varsigma} := \frac{1}{l_n} \sum_{i\in\widehat{\mathbb{I}}^n_{t,\varsigma}} \Gamma_{t,\varsigma}\left(\chi^2_i - 1\right), \; H^n_{t,\varsigma} := \frac{1}{l_n} \sum_{i\in\widehat{\mathbb{I}}^n_{t,\varsigma}} \left(\Gamma^n_i - \Gamma_{t,\varsigma}\right), \; D^n_{t,\varsigma} := \frac{1}{l_n} \sum_{i\in\widehat{\mathbb{I}}^n_{t,\varsigma}} (\Gamma^n_i - \Gamma_{t,\varsigma})(\chi^2_i - 1). \tag{SA.6}$$

The following theorem gives the joint stable limit of the two leading terms.

Theorem SA.1. *Let Assumption (S-HONV) hold, and further assume Condition A.2 and $c_n := l^2_n\Delta_n \to c \in [0,\infty]$. We have*

$$u_n \left(G^n_{t,\varsigma}, H^n_{t,\varsigma}\right)_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}} \xrightarrow{\mathcal{L}_s} \left(G_{t,\varsigma}, H_{t,\varsigma}\right)_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}}, \tag{SA.7}$$

where $(G_{t,\varsigma})_{(t,\varsigma)}$ *and* $(H_{t,\varsigma})_{(t,\varsigma)}$*, defined on an extension of* $(\Omega, \mathcal{F}, \mathbb{P})$*, are jointly centered Gaussian conditionally on* $\mathcal{F}$ *with the following covariance structure: for* $t, t' \in \mathcal{T}$ *and* $\varsigma, \varsigma' \in \{+,-\}$*,*

$$\widetilde{\mathbb{E}}\big(G_{t,\varsigma}G_{t',\varsigma'}|\mathcal{F}\big) = \frac{\Gamma_{t,\varsigma}^2 \mathfrak{S}\mathbf{1}_{\{\varsigma=\varsigma',t=t'\}}}{1+c}, \quad \widetilde{\mathbb{E}}\big(H_{t,\varsigma}H_{t',\varsigma'}|\mathcal{F}\big) = \frac{c\widetilde{\sigma}_{t,\varsigma}^2 \mathbf{1}_{\{\varsigma=\varsigma',t=t'\}}}{3\alpha_{t,\varsigma}(1+c)}, \quad \widetilde{\mathbb{E}}\big(G_{t,\varsigma}H_{t',\varsigma'}|\mathcal{F}\big) = 0. \tag{SA.8}$$

Theorem SA.1 is a consequence of Lemma SA.2, Lemma SA.3 below, and Lemmas SA.8 to SA.10 in Section SA.4. The next lemma replaces $H_{t,\varsigma}^n$ by a Brownian average and shows that $D_{t,\varsigma}^n$ is negligible.

Lemma SA.3. *Let the assumptions of Theorem SA.1 hold. If* $v > 1$*,* $l_n \to \infty$ *and* $l_n\Delta_n \to 0$*, for any* $t \in \mathcal{T}$ *and* $\varsigma \in \{+,-\}$*, we have* $u_n(\frac{\widetilde{\sigma}_{t,\varsigma}}{\sqrt{\alpha_{t,\varsigma}}}\widetilde{H}_{t,\varsigma}^n - H_{t,\varsigma}^n) \xrightarrow{\mathbb{P}} 0$ *and* $u_n D_{t,\varsigma}^n \xrightarrow{\mathbb{P}} 0$*, where*

$$\widetilde{H}_{t,\varsigma}^n := \frac{\sqrt{\alpha_{t,\varsigma}}}{l_n} \sum\nolimits_{i \in \widehat{\mathbb{I}}_{t,\varsigma}^n} \left(W_{t_i^n} - W_t\right). \tag{SA.9}$$

The remaining three lemmas compare the feasible estimator $\widehat{\Gamma}_{t,\varsigma}^n$ with the oracle estimator $\widetilde{\Gamma}_{t,\varsigma}^n$ through two intermediate statistics.

Lemma SA.4. *Let Assumption (S-HONV) hold, and further assume* $(\overline{k}_n \vee l_n)\Delta_n \to 0$ *and* $\overline{k}_n/l_n \to 0$*. Then, for* $\varsigma \in \{+,-\}$*, we have* $u_n(\widehat{\Gamma}_{t,\varsigma}^n - \overline{\Gamma}_{t,\varsigma}^n) \xrightarrow{\mathbb{P}} 0$*, where* $\overline{\Gamma}_{t,\varsigma}^n := \frac{1}{l_n}\sum_{i \in \mathbb{I}_{t,\varsigma}^n} \Gamma_i^n \overline{f}(\chi)_i^n$*.*

Lemma SA.5. *Let Assumption (S-HONV) hold, and* $v > 1$*. For any* $\varsigma \in \{+,-\}$*, we have* $\mathbb{E}\Big((\widetilde{\Gamma}_{t,\varsigma}^{\mathrm{tr},n} - \overline{\Gamma}_{t,\varsigma}^n)^2\Big) \le K\left(\frac{1}{p_n l_n} \vee k_n^{-v}\right)$*, where the trimmed oracle statistic is* $\widetilde{\Gamma}_{t,\varsigma}^{\mathrm{tr},n} := \frac{1}{l_n}\sum_{i \in \mathbb{I}_{t,\varsigma}^n} \Gamma_i^n \chi_i^2$*.*

Lemma SA.6. *Let Assumption (S-HONV) hold. For any* $\varsigma \in \{+,-\}$*, we have* $\mathbb{E}\Big(|\widetilde{\Gamma}_{t,\varsigma}^n - \widetilde{\Gamma}_{t,\varsigma}^{\mathrm{tr},n}|\Big) \le K\overline{k}_n/l_n$*. In particular, if* $u_n\overline{k}_n/l_n \to 0$*, then* $u_n(\widetilde{\Gamma}_{t,\varsigma}^n - \widetilde{\Gamma}_{t,\varsigma}^{\mathrm{tr},n}) \xrightarrow{\mathbb{P}} 0$*.*

SA.3 Proofs of the main results

Proof of Theorem 3.1. For $(t,\varsigma) \in \mathcal{T} \times \{+,-\}$, recall the decomposition $\widetilde{\Gamma}_{t,\varsigma}^n - \Gamma_{t,\varsigma} = G_{t,\varsigma}^n + H_{t,\varsigma}^n + D_{t,\varsigma}^n$ in (SA.6). Conditioning first on $\mathcal{F}^{(0)}$, by (SA.1) and the boundedness of Γ, we have $\mathbb{E}((G_{t,\varsigma}^n)^2 \mid \mathcal{F}^{(0)}) \le Kl_n^{-2}\sum_{i,i' \in \widehat{\mathbb{I}}_{t,\varsigma}^n}(1+|i-i'|)^{-v} \le Kl_n^{-1}$ as $v > 1$. Hence, $G_{t,\varsigma}^n \xrightarrow{\mathbb{P}} 0$ as $l_n \to \infty$.

By Lemma SA.1 and Lemma SA.2, and the one-sided continuity of Γ, we have $e_{t,\varsigma}^n := \max_{i \in \widehat{\mathbb{I}}_{t,\varsigma}^n} |\Gamma_i^n - \Gamma_{t,\varsigma}| \xrightarrow{\mathbb{P}} 0$. Since $H_{t,\varsigma}^n + D_{t,\varsigma}^n = \frac{1}{l_n}\sum_{i \in \widehat{\mathbb{I}}_{t,\varsigma}^n}(\Gamma_i^n - \Gamma_{t,\varsigma})\chi_i^2$, and this average of χ_i^2 has conditional expectation one given $\mathcal{F}^{(0)}$, we get $|H_{t,\varsigma}^n + D_{t,\varsigma}^n| \le e_{t,\varsigma}^n O_P(1) \xrightarrow{\mathbb{P}} 0$. Thus, we have the convergence $\widetilde{\Gamma}_{t,\varsigma}^n \xrightarrow{\mathbb{P}} \Gamma_{t,\varsigma}$ for each (t,ς) under Condition A.1. Since $\mathcal{T} \times \{+,-\}$ is finite, these convergence results hold jointly. □

Proof of Theorem 3.2. By (SA.6) and Lemma SA.3, $u_n(\widetilde{\Gamma}_{t,\varsigma}^n - \Gamma_{t,\varsigma}) = u_n(G_{t,\varsigma}^n + H_{t,\varsigma}^n) + o_P(1)$ jointly over the finite set $\mathcal{T} \times \{+,-\}$. Hence Theorem SA.1 gives the joint limit $Z_{t,\varsigma} := G_{t,\varsigma} + H_{t,\varsigma}$, which

by (SA.8) is, conditionally on $\mathcal{F}$, centered Gaussian with independent coordinates and variances $\mathfrak{S}_{t,\varsigma}$. □

Proof of Theorem 3.3. Fix $(t,\varsigma) \in \mathcal{T} \times \{+,-\}$. By (SA.6), $\widehat{\Gamma}^n_{t,\varsigma} - \Gamma_{t,\varsigma} = G^n_{t,\varsigma} + H^n_{t,\varsigma} + D^n_{t,\varsigma} + \sum_{\ell=1}^3 \mathfrak{R}(\ell)^n_{t,\varsigma}$, where $\mathfrak{R}(1)^n_{t,\varsigma} := \widehat{\Gamma}^n_{t,\varsigma} - \overline{\Gamma}^n_{t,\varsigma}$, $\mathfrak{R}(2)^n_{t,\varsigma} := \overline{\Gamma}^n_{t,\varsigma} - \widetilde{\Gamma}^{\mathrm{tr},n}_{t,\varsigma}$, and $\mathfrak{R}(3)^n_{t,\varsigma} := \widetilde{\Gamma}^{\mathrm{tr},n}_{t,\varsigma} - \widetilde{\Gamma}^n_{t,\varsigma}$, with $\overline{\Gamma}^n_{t,\varsigma}$ and $\widetilde{\Gamma}^{\mathrm{tr},n}_{t,\varsigma}$ defined in Lemmas SA.4 and SA.5. Recall that $\overline{k}_n = k_n + p_n \asymp \Delta_n^{-(\kappa \vee \varrho)}$ and $u_n^{-2} = l_n^{-1} + l_n \Delta_n$.

Under Condition A.3, we have $(\overline{k}_n \vee l_n)\Delta_n \to 0$, $\overline{k}_n / l_n \to 0$, and $k_n, l_n, u_n \to \infty$. Hence, Lemma SA.4 gives $\mathfrak{R}(1)^n_{t,\varsigma} \xrightarrow{\mathbb{P}} 0$, Lemma SA.5 gives $\mathbb{E}\big((\mathfrak{R}(2)^n_{t,\varsigma})^2\big) \le K((p_n l_n)^{-1} \vee k_n^{-v}) \to 0$, and Lemma SA.6 gives $\mathbb{E}\big(|\mathfrak{R}(3)^n_{t,\varsigma}|\big) \le K\overline{k}_n / l_n \to 0$. Since Condition A.3 implies Condition A.1, Theorem 3.1 gives $G^n_{t,\varsigma} + H^n_{t,\varsigma} + D^n_{t,\varsigma} \xrightarrow{\mathbb{P}} 0$, and (8) follows.

Under Condition A.4, we have $u_n \asymp \Delta_n^{-(\lambda \wedge (1-\lambda))/2}$, and $\kappa \vee \varrho < \lambda - (\lambda \wedge (1-\lambda))/2$ gives $u_n \overline{k}_n / l_n \to 0$. Hence, Lemmas SA.4 and SA.6 give $u_n \mathfrak{R}(1)^n_{t,\varsigma} \xrightarrow{\mathbb{P}} 0$ and $u_n \mathfrak{R}(3)^n_{t,\varsigma} \xrightarrow{\mathbb{P}} 0$. Since $u_n^2 \le l_n$, Lemma SA.5 gives $u_n^2 \mathbb{E}\big((\mathfrak{R}(2)^n_{t,\varsigma})^2\big) \le K(p_n^{-1} \vee u_n^2 k_n^{-v}) \to 0$, as $\varrho > 0$ and $v\kappa > \lambda \wedge (1-\lambda)$. Therefore,

$$u_n(\widehat{\Gamma}^n_{t,\varsigma} - \widetilde{\Gamma}^n_{t,\varsigma}) = u_n \sum\nolimits_{\ell=1}^3 \mathfrak{R}(\ell)^n_{t,\varsigma} \xrightarrow{\mathbb{P}} 0, \tag{SA.10}$$

which is the oracle equivalence. Since Condition A.4 implies Condition A.2 and $\mathcal{T} \times \{+,-\}$ is finite, Theorem 3.2 and (SA.10) give the joint stable convergence with $\widehat{\Gamma}^n_{t\pm}$ in place of $\widetilde{\Gamma}^n_{t\pm}$.

Finally, Theorem B.1 gives $\widehat{\mathfrak{S}}^n_{t,\varsigma} \xrightarrow{\mathbb{P}} \mathfrak{S}_{t,\varsigma}$, and $\mathfrak{S}_{t,\varsigma}$ is $\mathcal{F}_\infty$-measurable. Since $\mathfrak{S}_{t,\varsigma} > 0$ almost surely by assumption, $\mathbb{P}(\widehat{\mathfrak{S}}^n_{t,\varsigma} \le 0) \to 0$, so defining the ratio in (9) to be zero on this event does not affect its limit. As the stable limit of $u_n(\widehat{\Gamma}^n_{t,\varsigma} - \Gamma_{t,\varsigma})$ is, conditionally on $\mathcal{F}$, centered Gaussian with variance $\mathfrak{S}_{t,\varsigma}$, stable Slutsky convergence gives (9). □

Proof of Corollary 3.1. By Theorem 3.3 and the conditional independence in Theorem 3.2, we have the following stable convergence: $u_n(\widehat{\Gamma}^n_{t+} - \widehat{\Gamma}^n_{t-} - \Delta\Gamma_t) \xrightarrow{\mathcal{L}_s} Z_{t+} - Z_{t-}$, which is, conditionally on $\mathcal{F}$, centered Gaussian with variance V_t. Moreover, Theorem B.1 gives $V^n_t \xrightarrow{\mathbb{P}} V_t$, where V_t is $\mathcal{F}_\infty$-measurable and a.s. positive. Consequently, $\mathbb{P}(V^n_t \le 0) \to 0$. Suppose $\mathbb{P}(\Delta\Gamma_t = 0) > 0$; otherwise the first conclusion is immediate. Since $\{\Delta\Gamma_t = 0\} \in \mathcal{F}_t \subseteq \mathcal{F}_\infty$, the stable convergence yields $T^n_t \xrightarrow{\mathcal{L}_s} \mathcal{N}(0,1)$, and this proves (12). Since $T^n_t / u_n \xrightarrow{\mathbb{P}} \widetilde{T}_t := \Delta\Gamma_t / \sqrt{V_t}$, for any $\epsilon > 0$ and large n, $\mathbb{P}(\Delta\Gamma_t \ne 0,\ |T^n_t| \le z_{1-a/2}) \le \mathbb{P}(|T^n_t/u_n - \widetilde{T}_t| > \epsilon) + \mathbb{P}(0 < |\widetilde{T}_t| \le 2\epsilon)$. Letting $n \to \infty$ and then $\epsilon \downarrow 0$ proves (13). □

SA.4 Proof of Technical Lemmas

Lemma SA.7. *Let Assumption (N) hold. For a fixed nonnegative integer m and $i \in \mathbb{Z}$, define $\zeta_i := Q_i(\chi_i, \ldots, \chi_{i+m})$ where Q_i are measurable functions such that $\mathbb{E}(\zeta_i) = 0$ and $\sup_i \|\zeta_i\|_p < \infty$ for any $p \ge 2$. Then, for any $p \ge 2$ and any finite set $I \subset \mathbb{Z}$, $\mathbb{E}\big(\big|\sum_{i \in I} \zeta_i\big|^p\big) \le K|I|^{p/2}$.*

Proof. The sequence $\{\zeta_i\}_i$ is ρ-mixing with coefficients $\rho^\zeta_k \le \rho_{(k-m)\vee 0}$ since $\sigma(\zeta_i : i \le a) \subset \mathcal{G}_{a+m}$ and $\sigma(\zeta_i : i \ge a+k) \subset \mathcal{G}^{a+k}$. We denote the elements in I as $I \equiv \{i_\ell\}_{1 \le \ell \le |I|}$ in increasing order. As we have $i_{\ell+r} - i_\ell \ge r$ for $1 \le \ell < \ell + r \le |I|$, the reordered sequence $\{\zeta_{i_\ell}\}_{1 \le \ell \le |I|}$ is also ρ-mixing with coefficients bounded by ρ^ζ_k. Under Assumption (N), since m is fixed, we have

$\sum_{r\geq 0}(\rho^{\zeta}_{2^r})^{2/p} \leq \sum_{r\geq 0}\rho^{2/p}_{(2^r-m)\vee 0} < \infty$ for any $p \geq 2$ and $v > 0$. Then, we obtain $\mathbb{E}(|\sum_{\ell\leq|I|}\zeta_{i_\ell}|^p) \leq K(|I|^{p/2}\sup_i\|\zeta_i\|_2^p + |I|\sup_i\|\zeta_i\|_p^p) \leq K|I|^{p/2}$ following Theorem 1.1 of Shao (1995). The proof is complete. □

Proof of Lemma SA.1. For $s \in (0,1)$ and $k \in \mathbb{N}$, let $I(k)^n_s := (s-(k+1)\Delta_n,\, s-k\Delta_n]$. For $m \geq 1$, let $N(k,m)^n_s := \sum_{i=1}^m \mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}}$, and let $\Delta(k,m)^{n*}_s$ be the last corresponding duration, or zero when the count is zero. Since $\{t^n_{i-1} \in I(k)^n_s\} \in \mathcal{I}^n_{i-1}$ and α is bounded by Assumption (S-HONV), (3) gives $\mathbb{E}\left(\Delta(n,i)|\,\mathcal{I}^n_{i-1}\right) \geq K\Delta_n$ for large n, and we have

$$\sum_{i=1}^m \mathbb{E}\Big(\Delta(n,i)\mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}}\Big) = \sum_{i=1}^m \mathbb{E}\Big(\mathbb{E}\left(\Delta(n,i)|\,\mathcal{I}^n_{i-1}\right)\mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}}\Big) \geq K\Delta_n\mathbb{E}(N(k,m)^n_s).$$

By definition, we also have $\sum_{i=1}^m \Delta(n,i)\mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}} \leq \Delta_n + \Delta(k,m)^{n*}_s$. Thus,

$$K\Delta_n\mathbb{E}(N(k,m)^n_s) \leq \Delta_n + \mathbb{E}(\Delta(k,m)^{n*}_s). \tag{SA.11}$$

Moreover, $(\Delta(k,m)^{n*}_s)^2 \leq \sum_{i=1}^m \Delta(n,i)^2\mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}}$, so Assumption (S-HONV) and successive conditioning imply $\mathbb{E}\big((\Delta(k,m)^{n*}_s)^2\big) \leq K\Delta_n^2\mathbb{E}(N(k,m)^n_s)$, which combined with the Cauchy–Schwarz (CS) inequality yields $\mathbb{E}(\Delta(k,m)^{n*}_s) \leq K\Delta_n\sqrt{\mathbb{E}(N(k,m)^n_s)}$. This estimate combined with (SA.11) leads to $K\Delta_n\mathbb{E}(N(k,m)^n_s) \leq \Delta_n + K\Delta_n\sqrt{\mathbb{E}(N(k,m)^n_s)}$. Thus, $\mathbb{E}(N(k,m)^n_s) \leq K$, uniformly in s,k,m. Monotone convergence yields $\mathbb{E}(N(k)^n_s) \leq K$ for $N(k)^n_s := \sum_{i\geq 1}\mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}}$.

The intervals $I(k)^n_s$ partition $(-\infty, s]$. If the duration containing s starts in $I(k)^n_s$, it exceeds $k\Delta_n$, since $t^n_{n_s} \leq s < t^n_{n_s+1}$. For $k \geq 1$, we have the estimate for the conditional moment that $\mathbb{E}\big(\Delta(n,i)\mathbf{1}_{\{\Delta(n,i)>k\Delta_n\}} \mid \mathcal{I}^n_{i-1}\big) \leq (k\Delta_n)^{-2}\mathbb{E}\big(\Delta(n,i)^3 \mid \mathcal{I}^n_{i-1}\big) \leq K\Delta_n k^{-2}$, which yields by successive conditioning that

$$\mathbb{E}(\Delta(n,n_s+1)) \leq \sum_{k\geq 0}\sum_{i\geq 1}\mathbb{E}\Big(\mathbf{1}_{\{t^n_{i-1}\in I(k)^n_s\}}\Delta(n,i)\mathbf{1}_{\{\Delta(n,i)>k\Delta_n\}}\Big) \leq K\Delta_n\sum_{k\geq 0}\frac{\mathbb{E}(N(k)^n_s)}{(1+k)^2} \leq K\Delta_n.$$

For $r \geq 2$, the event $\{n_s = j\}$ belongs to $\mathcal{I}^n_{j+1} \subseteq \mathcal{I}^n_{j+r-1}$, so conditioning and summing over $j \geq 0$ give $\mathbb{E}(\Delta(n,n_s+r)) \leq K\Delta_n$. Hence

$$\mathbb{E}\big(t^n_{n_s+\ell} - t^n_{n_s}\big) = \mathbb{E}(\Delta(n,n_s+1)) + \sum_{r=2}^{\ell}\mathbb{E}(\Delta(n,n_s+r)) \leq K\ell\Delta_n.$$

This proves (SA.2). Since $0 \leq t^n_{n_s+\ell} - s \leq t^n_{n_s+\ell} - t^n_{n_s}$, Markov's inequality gives (SA.3). □

In the subsequent proof, we also use Γ_t as the shorthand notation for $\Gamma_{t,+}$. Next, we use a construction of auxiliary index sets for proving the properties of all pre-t estimators. This construction yields index sets that start from a valid stopping time and differ only negligibly from the original index sets. The constants $\underline{\alpha}$, $\overline{\alpha}$ and C, the sequences j_n and h_n, the index a_n, and J^n_t, $\widetilde{J}^n_t$ are defined before Lemma SA.2 in Section SA.2.

Proof of Lemma SA.2. Fix $t \in \mathcal{T}$, and let $s_n := t - h_n > 0$. Since $\{a_n \le i\} = \{t_i^n > s_n\} \in \mathcal{I}_i^n$, a_n is a stopping index and $t_{a_n}^n$ is a stopping time. For $r \ge 0$, let $\mathcal{H}_r^n = \mathcal{I}_{a_n+r}^n$, and define, for $r \ge 1$,

$$\mathfrak{A}(1)_r^n := \mathbb{E}\left(\Delta(n, a_n + r) \middle| \mathcal{H}_{r-1}^n\right) - \frac{\Delta_n}{\alpha_{a_n+r-1}^n}, \mathfrak{A}(2)_r^n := \Delta(n, a_n + r) - \mathbb{E}\left(\Delta(n, a_n + r) \middle| \mathcal{H}_{r-1}^n\right).$$

Since $\{a_n = k\} \in \mathcal{I}_k^n \subseteq \mathcal{I}_{k+r-1}^n$, then $\mathfrak{A}(1)_r^n = \sum_{k\ge1} \mathbf{1}_{\{a_n=k\}} \left(\mathbb{E}\left(\Delta(n, k + r) \middle| \mathcal{I}_{k+r-1}^n\right) - \frac{\Delta_n}{\alpha_{k+r-1}^n}\right)$. Since k and r are deterministic, Assumption (S-HONV) implies that $|\mathfrak{A}(1)_r^n| \le K\Delta_n^{1+\eta}$. A similar decomposition gives $\mathbb{E}\left((\mathfrak{A}(2)_r^n)^2 \middle| \mathcal{H}_{r-1}^n\right) \le K\Delta_n^2$. This bound, together with the martingale difference property of $\mathfrak{A}(2)_r^n$ with respect to $(\mathcal{H}_r^n)$, gives $\mathbb{E}\left(\max_{0\le j\le m} \left|\sum_{r=1}^j \mathfrak{A}(2)_r^n\right|\right) \le K\sqrt{m}\,\Delta_n$ after an application of Lemma A.6 of Jacod et al. (2017).

Choose a fixed $L > C\overline{\alpha}$ and set $j_n' = \lceil Lj_n \rceil$. Let $\mathcal{J}_t^n := \{J_t^n < j_n'\}$ and $\mathcal{J}_t'^n := \{t_{a_n}^n < t\}$. On their intersection, $0 \le J_t^n < j_n'$ and all the relevant observation times lie in $[s_n, t]$. Let $\overline{J}_t^n := C\alpha_{s_n} j_n$ and $\mathfrak{B}_t^n := (t - t_{n_t}^n) + (t_{a_n}^n - s_n)$. Note that $\mathfrak{B}_t^n \ge 0$. Summing the durations from a_n to n_t gives

$$\frac{J_t^n - \widetilde{J}_t^n}{j_n} = \frac{\overline{J}_t^n - \widetilde{J}_t^n}{j_n} - \frac{\alpha_{s_n}}{j_n} \sum_{r=1}^{J_t^n} \left(\frac{1}{\alpha_{a_n+r-1}^n} - \frac{1}{\alpha_{s_n}}\right) - \frac{\alpha_{s_n}}{j_n\Delta_n} \left\{\sum_{r=1}^{J_t^n} \left(\mathfrak{A}(1)_r^n + \mathfrak{A}(2)_r^n\right) + \mathfrak{B}_t^n\right\}.$$

Here $1 \le \overline{J}_t^n - \widetilde{J}_t^n < 2$, and (SA.2) at t and s_n gives $\mathbb{E}(\mathfrak{B}_t^n) \le K\Delta_n$. We also have the regular bound $\mathbb{E}\left(\sup_{u\in[s_n,t]} |1/\alpha_u - 1/\alpha_{s_n}|\right) \le K\sqrt{h_n}$. This, together with the preceding identity, and the bounds above for $\mathfrak{A}(1)_r^n$ and $\mathfrak{A}(2)_r^n$, yields

$$\mathbb{E}\left(\mathbf{1}_{\{\mathcal{J}_t^n \cap \mathcal{J}_t'^n\}} \left|(J_t^n - \widetilde{J}_t^n)/j_n\right|\right) \le K\left(\sqrt{h_n} \vee \Delta_n^\eta \vee j_n^{-1/2}\right) \to 0.$$

It remains to remove the two event restrictions. On $(\mathcal{J}_t^n)^c$, $t_{a_n+j_n'}^n \le t$, so $\sum_{r=1}^{j_n'} \Delta(n, a_n + r) \le h_n$. All the starting times in this sum lie in (s_n, t), and hence

$$\frac{1}{j_n'\Delta_n} \sum_{r=1}^{j_n'} \left(\mathfrak{A}(1)_r^n + \mathfrak{A}(2)_r^n\right) \le \frac{Cj_n}{j_n'} - \frac{1}{\overline{\alpha}} \le -\left(\frac{1}{\overline{\alpha}} - \frac{C}{L}\right).$$

The last parenthesis is a fixed positive constant. The bounds above for $\mathfrak{A}(1)_r^n$ and $\mathfrak{A}(2)_r^n$, followed by Markov's inequality, give $\mathbb{P}((\mathcal{J}_t^n)^c) \le K_L(\Delta_n^\eta \vee j_n^{-1/2}) \to 0$. Also, $(\mathcal{J}_t'^n)^c = \{t_{a_n}^n - s_n \ge h_n\}$, so (SA.3) gives $\mathbb{P}((\mathcal{J}_t'^n)^c) \le K\Delta_n/h_n \le K/j_n \to 0$. This proves (SA.4).

Since $\mathcal{T}$ is finite, (SA.4) also holds with the maximum of the absolute errors over $t \in \mathcal{T}$. A standard diagonal argument, using $j_n \to \infty$, gives deterministic integers $b_n \to \infty$ with $b_n/j_n \to 0$ and $\mathbb{P}(\max_{t\in\mathcal{T}} |J_t^n - \widetilde{J}_t^n| > b_n) \to 0$. This proves (SA.5). □

Let $a_t^n := a_n + (\widetilde{J}_t^n - b_n)^+$ and $\mathcal{B}_t^n := \{t_{a_n}^n < t,\ \widetilde{J}_t^n \ge j_n + b_n\}$. On $\mathcal{B}_t^n$, define the forward auxiliary sets $\mathbb{I}^\circ(j_n)_{t,-}^n := \{a_t^n - j_n + \overline{k}_n, \ldots, a_t^n - \overline{k}_n - 1\}$, $\widehat{\mathbb{I}}^\circ(j_n)_{t,-}^n := \{a_t^n - j_n, \ldots, a_t^n - 1\}$, and set both sets equal to $\emptyset$ on $(\mathcal{B}_t^n)^c$. For the trimmed set $\mathbb{I}^\circ(j_n)_{t,-}^n$, we require $j_n \ge 2\overline{k}_n + 1$, as holds in all its subsequent uses. These sets are $\mathcal{I}_{a_n}^n$-measurable and contain only indices at least a_n. Under the

assumptions of Lemma SA.2, we have $\mathcal{B}_t^n \in \mathcal{I}_{a_n}^n$ and $\mathcal{A}_t^n \in \mathcal{F}_t^n$. Both follow from the definitions. Since $C\underline{\alpha} > 2$ and $b_n/j_n \to 0$, we have $\widetilde{J}_t^n \geq j_n + b_n$ for all sufficiently large n. On $\mathcal{A}_t^n$, this gives $J_t^n \geq j_n$, hence $t_{a_n}^n < t$ and $\mathcal{A}_t^n \subseteq \mathcal{B}_t^n$. On $\mathcal{A}_t^n$, we then have $0 \leq n_t - a_t^n = J_t^n - \widetilde{J}_t^n + b_n \leq 2b_n$, and all observation times indexed by the auxiliary sets lie in $(t - h_n, t)$.

Let $g_0 := \min_{t \neq t' \in \mathcal{T}} |t - t'| > 0$, with the convention $g_0 := 1$ if $|\mathcal{T}| = 1$. For $t \in \mathcal{T}$, let $\tau_t := (t + g_0/3) \wedge 1$ and $\underline{\tau}_t := (t - g_0/3) \vee 0$. Let r_n be a deterministic sequence of positive integers such that $r_n \to \infty$ and $r_n \Delta_n \to 0$, and define $\Omega(r_n)_t^n := \{t_{n_t + r_n}^n < \tau_t,\ t_{n_t - r_n}^n > \underline{\tau}_t\}$. Under Assumption (S-HONV),

$$\mathbb{P}\left(\bigcap_{t \in \mathcal{T}} \Omega(r_n)_t^n\right) \to 1. \tag{SA.12}$$

Indeed, apply Lemma SA.2 with $j_n = r_n$ and $h_n = C r_n \Delta_n$. Since $\widetilde{J}_t^n / r_n \geq C\underline{\alpha} - 2/r_n$, (SA.4) implies $\mathbb{P}(J_t^n < r_n) \to 0$. On $\{J_t^n \geq r_n\}$, we have $t_{n_t - r_n}^n \geq t_{a_n}^n > t - h_n > \underline{\tau}_t$ eventually. Together with (SA.3), this yields $\mathbb{P}\left((\Omega(r_n)_t^n)^c\right) \leq \frac{K r_n \Delta_n}{\tau_t - t} + \mathbb{P}(J_t^n < r_n) \to 0$. Then (SA.12) follows because $\mathcal{T}$ is finite.

Next, we turn to Theorem SA.1, which is stated in Section SA.2 together with the decomposition (SA.6). It is an immediate consequence of Lemma SA.2, proved above, Lemma SA.3, proved below, and Lemmas SA.8 to SA.10, which we state and prove below.

Lemma SA.8. *Let Assumption (S-HONV) hold. Assume $l_n \to \infty$, and $l_n \Delta_n \to 0$ as $n \to \infty$. For any $t \in \mathcal{T}$, and $\varsigma \in \{+, -\}$, we have $u_n(H_{t,\varsigma}^n - H_{t,\varsigma}'^n) \xrightarrow{\mathbb{P}} 0$, where $H_{t,\varsigma}'^n := \frac{1}{l_n} \sum_{i \in \widehat{\mathbb{I}}_{t,\varsigma}^n} \int_t^{t_i^n} \widetilde{\sigma}_u \mathrm{d}W_u$.*

Proof. Fix $t \in \mathcal{T}$. We consider the following proof on the event $\Omega(l_n)_t^n$ first, then extend to the general case. By (SA.12), we have $\mathbb{P}(\Omega(l_n)_t^n) \to 1$ for any $t \in \mathcal{T}$. Writing $\widetilde{b} := b^{\Gamma'}$ and $\widetilde{\vartheta} := \delta^{\Gamma'}$, we have

$$\Gamma_t' = \Gamma_0' + \int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + \left(\widetilde{\vartheta} \mathbf{1}_{\{|\widetilde{\vartheta}| \leq 1\}}\right) \star \left(\mathfrak{p} - \mathfrak{q}\right)_t + \left(\widetilde{\vartheta} \mathbf{1}_{\{|\widetilde{\vartheta}| > 1\}}\right) \star \mathfrak{p}_t.$$

It can also be represented as $\Gamma_t' = \Gamma_0' + \int_0^t \widetilde{b}_s' \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + M_t$ under Assumption (S-HONV), where the bounded process $\widetilde{b}_s'$ is defined as $\widetilde{b}_s' := \widetilde{b}_s + \int_{|\widetilde{\vartheta}| > 1} \widetilde{\vartheta}(s,z) \lambda(\mathrm{d}z)$, and $M := \widetilde{\vartheta} \star (\mathfrak{p} - \mathfrak{q})$. For any $\epsilon \in (0, 1]$, let $\widetilde{b}'(\epsilon)_s := \widetilde{b}_s' - \int_{\{J^{\Gamma'}(z) > \epsilon\}} \widetilde{\vartheta}(s,z) \lambda(\mathrm{d}z)$, $M(\epsilon)_t := \widetilde{\vartheta} \mathbf{1}_{\{J^{\Gamma'}(z) \leq \epsilon\}} \star (\mathfrak{p} - \mathfrak{q})_t$, and $J(\epsilon)_t := \widetilde{\vartheta} \mathbf{1}_{\{J^{\Gamma'}(z) > \epsilon\}} \star \mathfrak{p}_t$, where $J^{\Gamma'}(z)$ is the bounded deterministic envelope in Assumption (K). We can then rewrite $\Gamma_t' = \Gamma_0' + \int_0^t \widetilde{b}'(\epsilon)_s \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + M(\epsilon)_t + J(\epsilon)_t$.

Now on $\Omega(l_n)_t^n$ we make the decomposition $\Gamma_i^n - \Gamma_{t,\varsigma} - \int_t^{t_i^n} \widetilde{\sigma}_u \mathrm{d}W_u = \sum_{\ell=1}^3 \mathfrak{C}(\ell)_{i,\varsigma}^n$ for any $i \in \widehat{\mathbb{I}}_{t,\varsigma}^n$, where $\mathfrak{C}(1)_{i,+}^n := \int_t^{t_i^n} \widetilde{b}'(\epsilon)_s \mathrm{d}s$, $\mathfrak{C}(1)_{i,-}^n := -\int_{t_i^n}^t \widetilde{b}'(\epsilon)_s \mathrm{d}s$, $\mathfrak{C}(2)_{i,\varsigma}^n := M(\epsilon)_{t_i^n} - M(\epsilon)_{t,\varsigma}$, and $\mathfrak{C}(3)_{i,\varsigma}^n := J(\epsilon)_{t_i^n} - J(\epsilon)_{t,\varsigma}$.

Let $\phi(\epsilon) := \int_{\{J^{\Gamma'}(z) \leq \epsilon\}} J^{\Gamma'}(z)^2 \lambda(\mathrm{d}z)$. Since $\widetilde{b}'$ is bounded and $\int_{\{J^{\Gamma'}(z) > \epsilon\}} J^{\Gamma'}(z) \lambda(\mathrm{d}z) \leq \epsilon^{-1} \int J^{\Gamma'}(z)^2 \lambda(\mathrm{d}z)$ under Assumption (K), we have $|\widetilde{b}'(\epsilon)_s| \leq K/\epsilon$ for all $s \geq 0$, while dominated convergence theorem leads to $\phi(\epsilon) \to 0$ as $\epsilon \to 0$. By (SA.2), optional isometry, the compensation

formula and the CS inequality, we have

$$\mathbb{E}\big(|\mathfrak{C}(2)^n_{n_t+m,+}|\big) \le \mathbb{E}\left(\int_t^{t^n_{n_t+m}} \int_{\{J^{\Gamma'}\le\epsilon\}} |\widetilde{\vartheta}(s,z)|^2 \lambda(\mathrm{d}z)\mathrm{d}s\right)^{1/2} \le K\sqrt{m\Delta_n\phi(\epsilon)},$$
$$\mathbb{E}\big(|\mathfrak{C}(1)^n_{n_t+m,+}| \vee |\mathfrak{C}(3)^n_{n_t+m,+}|\big) \le Km\Delta_n/\epsilon.$$

These estimates yield $\frac{u_n}{l_n}\sum_{m=1}^{l_n}\sum_{\ell=1}^{3}\mathbb{E}\big(|\mathfrak{C}(\ell)^n_{n_t+m,+}|\big) \le K\left(\sqrt{l_n\Delta_n}/\epsilon + \sqrt{\phi(\epsilon)}\right)$. Thus, for any fixed ϵ, we have $\limsup_{n\to\infty} u_n\mathbb{E}\Big(\mathbf{1}_{\{\Omega(l_n)^n_t\}}\big|H^n_{t,+} - H'^n_{t,+}\big|\Big) \le K\sqrt{\phi(\epsilon)}$. Since $\phi(\epsilon)\to 0$ as $\epsilon\to 0$, letting $\epsilon\to 0$ proves $\mathbb{E}\Big(\mathbf{1}_{\{\Omega(l_n)^n_t\}}u_n\big|H^n_{t,+} - H'^n_{t,+}\big|\Big)\to 0$. As $\mathbb{P}(\Omega(l_n)^n_t)\to 1$ for any $t\in\mathcal{T}$, we have

$$u_n\big|H^n_{t,+} - H'^n_{t,+}\big| \xrightarrow{\mathbb{P}} 0. \tag{SA.13}$$

For the pre-t estimator, we apply Lemma SA.2 with $j_n = l_n$. On $\mathcal{A}^n_t$, $J^n_t \ge l_n$ gives $n_t - l_n \ge a_n$, so $t - h_n < t^n_i < t$ for all $i\in\widehat{\mathbb{I}}^n_{t,-}$. Hence, pathwise on $\mathcal{A}^n_t\cap\Omega(l_n)^n_t$, for all such i, we have $|\mathfrak{C}(1)^n_{i,-}|\le Kh_n/\epsilon$ and

$$|\mathfrak{C}(2)^n_{i,-}| \le 2\sup_{s\in(t-h_n,t)}|M(\epsilon)_s - M(\epsilon)_{t-h_n}|, \quad |\mathfrak{C}(3)^n_{i,-}| \le \int_{t-h_n}^{t}\int_{\{J^{\Gamma'}>\epsilon\}} J^{\Gamma'}(z)\mathfrak{p}(\mathrm{d}s,\mathrm{d}z).$$

Applying the Burkholder–Davis–Gundy (BDG) inequality, we obtain

$$\frac{u_n}{l_n}\mathbb{E}\left(\mathbf{1}_{\{\mathcal{A}^n_t\cap\Omega(l_n)^n_t\}}\sum_{i\in\widehat{\mathbb{I}}^n_{t,-}}\sum_{\ell=1}^{3}|\mathfrak{C}(\ell)^n_{i,-}|\right) \le K\left(\sqrt{l_n\Delta_n}/\epsilon + \sqrt{\phi(\epsilon)}\right).$$

Since $\mathbb{P}(\mathcal{A}^n_t\cap\Omega(l_n)^n_t)\to 1$ by (SA.5) and (SA.12), the argument to obtain (SA.13) gives $u_n(H^n_{t,-} - H'^n_{t,-}) \xrightarrow{\mathbb{P}} 0$, which completes the proof. □

Proof of Lemma SA.3. Let $\mathfrak{r}(h)_{t,+} := \mathbb{E}\Big(\sup_{s\in[t,t+h]}|\widetilde{\sigma}_s - \widetilde{\sigma}_t|^2\Big)$, $\mathfrak{r}(h)_{t,-} := \mathbb{E}\Big(\sup_{s\in[t-h,t)}|\widetilde{\sigma}_s - \widetilde{\sigma}_{t-}|^2\Big)$ for any $0<h<\min(t,1-t)$. Let $\widetilde{H}'^n_{t,\varsigma} := \frac{1}{l_n}\sum_{i\in\widehat{\mathbb{I}}^n_{t,\varsigma}}\big(W_{t^n_i} - W_t\big)$. We begin with the first convergence. For the post-t part, we have $H'^n_{t,+} - \widetilde{\sigma}_t\widetilde{H}'^n_{t,+} = \frac{1}{l_n}\sum_{m=1}^{l_n}\int_t^{t^n_{n_t+m}}(\widetilde{\sigma}_s - \widetilde{\sigma}_t)\mathrm{d}W_s$. Let $\mathcal{C}(A)^n_t := \{t^n_{n_t+l_n} - t \le Al_n\Delta_n\}$ for fixed $A>0$. On $\mathcal{C}(A)^n_t$, the BDG inequality gives

$$\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{C}(A)^n_t\}}u_n^2\big|H'^n_{t,+} - \widetilde{\sigma}_t\widetilde{H}'^n_{t,+}\big|^2\Big) \le KAu_n^2l_n\Delta_n\,\mathfrak{r}(Al_n\Delta_n)_{t,+} \longrightarrow 0,$$

since $u_n^2l_n\Delta_n = c_n/(1+c_n)\le 1$. Moreover, we have $\lim_{A\to\infty}\limsup_n\mathbb{P}((\mathcal{C}(A)^n_t)^c) = 0$ by Lemma SA.1. Taking $n\to\infty$ first and then $A\to\infty$ proves the post-t part.

We now consider the pre-t part. Define the martingale $\overline{H}(u)_n := \int_{t-h_n}^{u}(\widetilde{\sigma}_s - \widetilde{\sigma}_{t-h_n})\mathrm{d}W_s$ for $u\in[t-h_n,t]$. By the BDG inequality, we have $\mathbb{E}\Big(\sup_{u\in[t-h_n,t]}|\overline{H}(u)_n|^2\Big) \le Kh_n\,\mathfrak{r}(h_n)_{t,-}$, and

$\mathbb{E}\Big(\sup_{u\in[t-h_n,t)}|W_u-W_{t-h_n}|^2\Big)\le Kh_n$. Then, on $\mathcal{A}_t^n$ (defined in Lemma SA.2), we have

$$H'^n_{t,-}-\widetilde{\sigma}_{t-}\widetilde{H}'^n_{t,-}=\frac{1}{l_n}\sum\nolimits_{i\in\widehat{\mathbb{I}}^n_{t,-}}\big(\overline{H}(t_i^n)_n-\overline{H}(t)_n\big)+(\widetilde{\sigma}_{t-h_n}-\widetilde{\sigma}_{t-})\widetilde{H}'^n_{t,-},$$

and $|\widetilde{H}'^n_{t,-}|\le 2\sup_{u\in[t-h_n,t)}|W_u-W_{t-h_n}|$. Since $u_n^2h_n\le K$ and $\mathbb{E}(|\widetilde{\sigma}_{t-h_n}-\widetilde{\sigma}_{t-}|^2)\le\mathfrak{r}(h_n)_{t,-}$, the two BDG bounds and the CS inequality give $\mathbb{E}(\mathbf{1}_{\{\mathcal{A}_t^n\}}u_n|H'^n_{t,-}-\widetilde{\sigma}_{t-}\widetilde{H}'^n_{t,-}|)\le K\sqrt{\mathfrak{r}(h_n)_{t,-}}\to 0$. Since $\mathbb{P}(\mathcal{A}_t^n)\to 1$ by (SA.5), it follows that $u_n(\widetilde{\sigma}_{t-}\widetilde{H}'^n_{t,-}-H'^n_{t,-})\xrightarrow{\mathbb{P}}0$. Combining this with the post-t part for $\varsigma=+$, and with Lemma SA.8 for the last term below, we obtain, for $\varsigma\in\{+,-\}$, $u_n\Big(\frac{\widetilde{\sigma}_{t,\varsigma}}{\sqrt{\alpha_{t,\varsigma}}}\widetilde{H}^n_{t,\varsigma}-H^n_{t,\varsigma}\Big)=u_n\big(\widetilde{\sigma}_{t,\varsigma}\widetilde{H}'^n_{t,\varsigma}-H'^n_{t,\varsigma}\big)+u_n\big(H'^n_{t,\varsigma}-H^n_{t,\varsigma}\big)\xrightarrow{\mathbb{P}}0$. This proves the first convergence in the lemma.

We now prove the second convergence. Let $\mathfrak{D}^n_{i,\varsigma}:=(\Gamma^n_i-\Gamma_{t,\varsigma})(\chi_i^2-1)$. For the post-$t$ part, we have

$$(D^n_{t,+})^2=\frac{1}{l_n^2}\sum\nolimits_{m=1}^{l_n}\big(\mathfrak{D}^n_{n_t+m,+}\big)^2+\frac{2}{l_n^2}\sum\nolimits_{k=1}^{l_n-1}\sum\nolimits_{m=1}^{l_n-k}\mathfrak{D}^n_{n_t+m,+}\mathfrak{D}^n_{n_t+m+k,+}.\tag{SA.14}$$

For $0\le k\le l_n-1$ and $1\le m\le l_n-k$, by the independence of $\mathcal{F}^{(0)}$ and $\mathcal{G}$, (SA.1), and the estimate (A.6) in Jacod et al. (2017) applied to Γ' at the forward stopping times $t^n_{n_t+m}$, we have $|\mathbb{E}\Big(\mathbf{1}_{\{\Omega(l_n)^n_t\}}\mathfrak{D}^n_{n_t+m,+}\mathfrak{D}^n_{n_t+m+k,+}\Big)|\le Kl_n\Delta_n(1+k)^{-v}$. Summing in (SA.14) and using $v>1$, we obtain $u_n^2\mathbb{E}\Big(\mathbf{1}_{\{\Omega(l_n)^n_t\}}(D^n_{t,+})^2\Big)\le Ku_n^2\Delta_n\le Kl_n\Delta_n\to 0$.

For the pre-t part, the boundedness of Γ and the BDG inequality, applied to Γ' on $[t-h_n,t)$, give $\mathbb{E}(\sup_{s\in[t-h_n,t)}|\Gamma_s-\Gamma_{t-}|^2)\le Kl_n\Delta_n$. On $\mathcal{A}_t^n$, this bounds $|\Gamma^n_i-\Gamma_{t,-}|$ uniformly over $i\in\widehat{\mathbb{I}}^n_{t,-}$, so the argument for the post-t part gives $u_n^2\mathbb{E}(\mathbf{1}_{\{\mathcal{A}_t^n\cap\Omega(l_n)^n_t\}}(D^n_{t,-})^2)\le Ku_n^2\Delta_n\to 0$. Since $\mathbb{P}(\mathcal{A}_t^n\cap\Omega(l_n)^n_t)\to 1$, Markov's inequality gives the second convergence in the lemma. □

Lemma SA.9. *Let the assumptions of Theorem SA.1 hold. Let* $\widetilde{G}^n_{t,\varsigma}:=\frac{1}{l_n}\sum_{i\in\widehat{\mathbb{I}}^n_{t,\varsigma}}(\chi_i^2-1)$ *for* $\varsigma\in\{+,-\}$. *We have*

$$u_n\left(\widetilde{G}^n_{t,\varsigma}\right)_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}}\xrightarrow{\mathcal{L}_s}(\widetilde{G}_{t,\varsigma})_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}},$$

where $(\widetilde{G}_{t,\varsigma})_{t,\varsigma}$ *is a centered Gaussian vector independent of* $\mathcal{F}$, *defined on an extension of the original probability space* $(\Omega,\mathcal{F},\mathbb{P})$. *It has the following conditional covariance structure: conditional on* $\mathcal{F}$, *for any* $t,t'\in\mathcal{T}$ *and* $\varsigma,\varsigma'\in\{+,-\}$, $\widetilde{\mathbb{E}}\Big(\widetilde{G}_{t,\varsigma}\widetilde{G}_{t',\varsigma'}|\mathcal{F}\Big)=\mathbf{1}_{\{t=t',\varsigma=\varsigma'\}}\frac{1}{1+c}\cdot\mathfrak{S}$, *where* $\mathfrak{S}$ *is defined in Theorem 3.2.*

Proof. We use the central limit theorem for ρ-mixing sequences in Corollary VIII.3.106 of Jacod and Shiryaev (2003); its process-level form treats the two adjacent windows at each $t\in\mathcal{T}$ jointly. Enumerate $\mathcal{T}$ as $t_1<\cdots<t_{|\mathcal{T}|}$, fix real coefficients $\theta=(\theta_{t,\varsigma})_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}}$, and write $\mathbf{n}=(n_t)_{t\in\mathcal{T}}$. Set $\widetilde{\chi}(i;\theta)_{t,\varsigma}:=\theta_{t,\varsigma}(\chi_i^2-1)$ and $\Xi_n:=u_n\sum_{(t,\varsigma)}\theta_{t,\varsigma}\widetilde{G}^n_{t,\varsigma}$, $\sigma_\theta^2:=\frac{\mathfrak{S}}{1+c}\sum_{(t,\varsigma)}\theta^2_{t,\varsigma}$. For every integer vector $\boldsymbol{\nu}=(\nu_t)_{t\in\mathcal{T}}$, define $\widehat{\mathbb{I}}(\boldsymbol{\nu})^n_{t,\pm}:=\{\nu_t\pm j:j=1,\dots,l_n\}$, $\Xi(\boldsymbol{\nu})'_n:=\frac{u_n}{l_n}\sum_{(t,\varsigma)}\sum_{i\in\widehat{\mathbb{I}}(\boldsymbol{\nu})^n_{t,\varsigma}}\widetilde{\chi}(i;\theta)_{t,\varsigma}$, and $\Phi_n(\boldsymbol{\nu};\theta):=\mathbb{E}(e^{i\Xi(\boldsymbol{\nu})'_n})$. Thus, $\Xi_n=\Xi(\mathbf{n})'_n$. It suffices to prove that Ξ_n converges $\mathcal{F}_\infty$-stably to a centered Gaussian variable with variance σ_θ^2 independent of $\mathcal{F}$.

Let $\zeta_i:=\chi_i^2-1$, so that $\widetilde{\chi}(i;\theta)_{t,\varsigma}=\theta_{t,\varsigma}\zeta_i$. Corollary VIII.3.106(iii) of Jacod and Shiryaev (2003),

with $p = 2$, only involves the law of $(\zeta_i)_{i\in\mathbb{Z}}$, so we may apply it on the canonical space of this sequence, with the shift as the measure-preserving map (see Appendix A.2 of Jacod et al. (2017)). It yields that the processes $X^m_s := m^{-1/2}\sum_{k=1}^{\lfloor ms\rfloor}\zeta_k$ converge finite-dimensionally in law to $\sqrt{\mathfrak{S}}\,W$ as $m\to\infty$, where W is a standard Brownian motion, because by stationarity the constant in (VIII.3.100) of Jacod and Shiryaev (2003) is $\mathbb{E}(\zeta_0^2) + 2\sum_{k\ge1}\mathbb{E}(\zeta_0\zeta_k) = \sum_{k\in\mathbb{Z}}(\mathbb{E}(\chi_0^2\chi_k^2) - 1) = \mathfrak{S}$. Since $l_n\to\infty$, we have $(X^{l_n}_1, X^{l_n}_2) \xrightarrow{\mathcal{L}} \sqrt{\mathfrak{S}}\,(W_1, W_2)$; see also (A.13) in Jacod et al. (2017).

For an integer ν, let $S(\nu)^n_- := \sum_{i=\nu-l_n}^{\nu-1}\zeta_i$, $S(\nu)^n_+ := \sum_{i=\nu+1}^{\nu+l_n}\zeta_i$, so that we have $\sum_{i\in\widehat{\mathbb{I}}(\boldsymbol{\nu})^n_{t,\varsigma}}\zeta_i = S(\nu_t)^n_\varsigma$. Shifting all indices by $l_n+1-\nu$ and using stationarity, we see that $(S(\nu)^n_-, S(\nu)^n_+)$ has the same law as

$$\Big(\sum\nolimits_{k=1}^{l_n}\zeta_k,\ \sum\nolimits_{k=l_n+2}^{2l_n+1}\zeta_k\Big) = \sqrt{l_n}\,\big(X^{l_n}_1,\ X^{l_n}_2 - X^{l_n}_1\big) + \big(0,\ \zeta_{2l_n+1} - \zeta_{l_n+1}\big).$$

The last vector is $o_p(\sqrt{l_n})$ since $\mathbb{E}(\zeta_0^2) = \mathbb{E}(\chi_0^4) - 1 < \infty$ under Assumption (N), and $u_n/\sqrt{l_n} = (1+c_n)^{-1/2}\to(1+c)^{-1/2}$. Since W_1 and $W_2 - W_1$ are independent standard normal variables, we obtain, for any $a_-, a_+\in\mathbb{R}$,

$$\psi_n(a_-,a_+) := \mathbb{E}\Big(\exp\Big(i\,\frac{u_n}{l_n}\big(a_-S(\nu)^n_- + a_+S(\nu)^n_+\big)\Big)\Big) \longrightarrow \exp\Big(-\frac{(a_-^2+a_+^2)\mathfrak{S}}{2(1+c)}\Big), \tag{SA.15}$$

where ψ_n does not depend on ν.

Let $\mathbb{A}_n := \{\boldsymbol{\nu} : \nu_{t_s} - \nu_{t_{s-1}} > 4l_n,\ 2\le s\le|\mathcal{T}|\}$. Recall g_0, τ_t, $\underline{\tau}_t$ and $\Omega(r_n)^n_t$ defined before (SA.12), and take $r_n = 2l_n$, so that $r_n\to\infty$ and $r_n\Delta_n\to0$. For $2\le s\le|\mathcal{T}|$, we have $\tau_{t_{s-1}} \le t_{s-1} + g_0/3 < t_s - g_0/3 \le \underline{\tau}_{t_s}$. Hence, on $\bigcap_{t\in\mathcal{T}}\Omega(2l_n)^n_t$, we have $t^n_{n_{t_{s-1}}+2l_n} < t^n_{n_{t_s}-2l_n}$ for all such s, that is, $\mathbf{n}\in\mathbb{A}_n$. Thus (SA.12) gives $\mathbb{P}(\mathbf{n}\in\mathbb{A}_n)\to1$.

Fix $\boldsymbol{\nu}\in\mathbb{A}_n$. For $1\le s\le|\mathcal{T}|$, let $U(\boldsymbol{\nu})^n_s := \frac{u_n}{l_n}\big(\theta_{t_s,-}S(\nu_{t_s})^n_- + \theta_{t_s,+}S(\nu_{t_s})^n_+\big)$, so that $\Xi(\boldsymbol{\nu})'_n = \sum_{s=1}^{|\mathcal{T}|}U(\boldsymbol{\nu})^n_s$, and $U(\boldsymbol{\nu})^n_s$ is a measurable function of $(\chi_i : |i-\nu_{t_s}|\le l_n)$. For $2\le r\le|\mathcal{T}|$, the variable $\prod_{s<r}e^{iU(\boldsymbol{\nu})^n_s}$ is $\mathcal{G}_{\nu_{t_{r-1}}+l_n}$-measurable, $e^{iU(\boldsymbol{\nu})^n_r}$ is $\mathcal{G}^{\nu_{t_r}-l_n}$-measurable, and $(\nu_{t_r}-l_n)-(\nu_{t_{r-1}}+l_n) > 2l_n$. Applying (SA.1) to the real and imaginary parts of these bounded variables, we have

$$\Big|\mathbb{E}\Big(\prod\nolimits_{s\le r}e^{iU(\boldsymbol{\nu})^n_s}\Big) - \mathbb{E}\Big(\prod\nolimits_{s<r}e^{iU(\boldsymbol{\nu})^n_s}\Big)\mathbb{E}\big(e^{iU(\boldsymbol{\nu})^n_r}\big)\Big| \le 4\rho_{2l_n} \le Kl_n^{-v}.$$

By stationarity, $\mathbb{E}(e^{iU(\boldsymbol{\nu})^n_s}) = \psi_n(\theta_{t_s,-},\theta_{t_s,+})$. Since all these characteristic functions have modulus at most 1, a telescoping argument over r gives

$$\sup_{\boldsymbol{\nu}\in\mathbb{A}_n}\Big|\Phi_n(\boldsymbol{\nu};\theta) - e^{-\sigma_\theta^2/2}\Big| \le K|\mathcal{T}|\,l_n^{-v} + \Big|\prod\nolimits_{s=1}^{|\mathcal{T}|}\psi_n(\theta_{t_s,-},\theta_{t_s,+}) - e^{-\sigma_\theta^2/2}\Big| \longrightarrow 0, \tag{SA.16}$$

where the convergence follows from (SA.15) and $\sigma_\theta^2 = \frac{1}{1+c}\sum_{s=1}^{|\mathcal{T}|}(\theta^2_{t_s,-} + \theta^2_{t_s,+})\mathfrak{S}$.

Finally, $\mathbf{n}$ is $\mathcal{F}^{(0)}$-measurable and χ is independent of $\mathcal{F}^{(0)}$, so conditioning on $\mathcal{F}^{(0)}$ fixes $\mathbf{n}$ and leaves the law of χ unchanged, as in the product structure of Appendix A.2 of Jacod et al. (2017).

Hence $\mathbb{E}(e^{i\Xi_n} \mid \mathcal{F}^{(0)}) = \Phi_n(\mathbf{n};\theta)$, and

$$\mathbb{E}\Big[\Big|\mathbb{E}(e^{i\Xi_n} \mid \mathcal{F}^{(0)}) - e^{-\sigma_\theta^2/2}\Big|\Big] \le \sup_{\boldsymbol{\nu}\in\mathbb{A}_n}\Big|\Phi_n(\boldsymbol{\nu};\theta) - e^{-\sigma_\theta^2/2}\Big| + 2\mathbb{P}(\mathbf{n}\notin\mathbb{A}_n) \longrightarrow 0, \tag{SA.17}$$

by (SA.16) and $\mathbb{P}(\mathbf{n}\in\mathbb{A}_n)\to 1$. Since $\mathcal{F}_\infty \subset \mathcal{F}^{(0)}$, it follows that $\mathbb{E}(Ve^{i\Xi_n}) \to \mathbb{E}(V)e^{-\sigma_\theta^2/2}$ for every bounded $\mathcal{F}_\infty$-measurable variable V. As θ is arbitrary, this proves the claimed $\mathcal{F}_\infty$-stable convergence. □

Lemma SA.10. *Let the assumptions of Theorem SA.1 hold. Let $\widetilde{H}^n_{t,\varsigma}$ and $\widetilde{G}^n_{t,\varsigma}$ be defined in (SA.9) and Lemma SA.9, respectively. We have*

$$u_n\left(\widetilde{H}^n_{t,\varsigma},\widetilde{G}^n_{t,\varsigma}\right)_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}} \xrightarrow{\mathcal{L}_s} (\widetilde{H}_{t,\varsigma},\widetilde{G}_{t,\varsigma})_{(t,\varsigma)\in\mathcal{T}\times\{+,-\}},$$

where $(\widetilde{H}_{t,\varsigma},\widetilde{G}_{t,\varsigma})_{t,\varsigma}$ is a centered Gaussian vector independent of $\mathcal{F}$, defined on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ of the original probability space $(\Omega,\mathcal{F},\mathbb{P})$. It has the following conditional covariance structure: conditional on $\mathcal{F}$, for any $t,t'\in\mathcal{T}$ and $\varsigma,\varsigma'\in\{+,-\}$,

$$\widetilde{\mathbb{E}}\Big(\widetilde{H}_{t,\varsigma}\widetilde{H}_{t',\varsigma'}|\mathcal{F}\Big) = \frac{c\mathbf{1}_{\{t=t',\varsigma=\varsigma'\}}}{1+c}\cdot\frac{1}{3},\quad \widetilde{\mathbb{E}}\Big(\widetilde{G}_{t,\varsigma}\widetilde{G}_{t',\varsigma'}|\mathcal{F}\Big) = \frac{\mathfrak{G}\mathbf{1}_{\{t=t',\varsigma=\varsigma'\}}}{1+c},\quad \widetilde{\mathbb{E}}\Big(\widetilde{H}_{t,\varsigma}\widetilde{G}_{t',\varsigma'}|\mathcal{F}\Big) = 0,$$

where $\mathfrak{G}$ is defined in Theorem 3.2.

Proof. Let $\mathbf{H}_n := u_n(\widetilde{H}^n_{t,\varsigma})_{(t,\varsigma)}$, $\mathbf{G}_n := u_n(\widetilde{G}^n_{t,\varsigma})_{(t,\varsigma)}$, $\Sigma_H := \frac{c}{3(1+c)}\mathbf{I}_{2|\mathcal{T}|}$ and $\Sigma_G := \frac{\mathfrak{G}}{1+c}\mathbf{I}_{2|\mathcal{T}|}$ where $\mathbf{I}_a$ denotes $a\times a$ identity matrix. Fix $x,y\in\mathbb{R}^{2|\mathcal{T}|}$ and a bounded $\mathcal{F}_\infty$-measurable variable V. Since x, y and V are arbitrary, it suffices to prove that $\mathbb{E}\big(Ve^{ix^*\mathbf{H}_n+iy^*\mathbf{G}_n}\big) \longrightarrow \mathbb{E}(V)\,e^{-(x^*\Sigma_H x+y^*\Sigma_G y)/2}$. Both V and $\mathbf{H}_n$ are $\mathcal{F}^{(0)}$-measurable, so (SA.17) with $\theta = y$ yields $\mathbb{E}(Ve^{ix^*\mathbf{H}_n+iy^*\mathbf{G}_n}) = e^{-y^*\Sigma_G y/2}\,\mathbb{E}(Ve^{ix^*\mathbf{H}_n}) + o(1)$. It thus remains to prove the display for $y=0$.

We now represent $x^*\mathbf{H}_n$, up to a negligible term, as a stochastic integral. We first define $\phi(s)^n_{t,+} := \frac{u_n\sqrt{\alpha_t}}{l_n}\sum_{i\in\widehat{\mathbb{I}}^n_{t,+}}\mathbf{1}_{\{t<s\le t^n_i\}}$ and $\phi(s)^n_{t,-} := -\mathbf{1}_{\{\mathcal{B}^n_t\}}\frac{u_n\sqrt{\alpha_{t-h_n}}}{l_n}\sum_{i\in\widehat{\mathbb{I}}^\circ(l_n)^n_{t,-}}\mathbf{1}_{\{t^n_i<s\le t\}}$. Note that both processes are predictable. Then, we have $u_n\widetilde{H}^n_{t,+} = \int_0^\infty\phi(s)^n_{t,+}\mathrm{d}W_s$. For the pre-$t$ part, we have $\mathbf{1}_{\{\mathcal{A}^n_t\}}\Big|u_n\widetilde{H}^n_{t,-} - \int_0^\infty\phi(s)^n_{t,-}\mathrm{d}W_s\Big| \le Ku_n\overline{W}^n_t\Big(\big|\sqrt{\alpha_{t-}}-\sqrt{\alpha_{t-h_n}}\big| + \frac{b_n}{l_n}\Big)$, where $\overline{W}^n_t := \sup_{u\in[t-h_n,t]}|W_u - W_{t-h_n}|$. The inequality is a consequence of the fact that on $\mathcal{A}^n_t$, all observation times indexed by $\widehat{\mathbb{I}}^n_{t,-}\cup\widehat{\mathbb{I}}^\circ(l_n)^n_{t,-}$ lie in $(t-h_n,t)$, and $|\widehat{\mathbb{I}}^n_{t,-}\triangle\widehat{\mathbb{I}}^\circ(l_n)^n_{t,-}| \le 4b_n$. $\mathbb{E}((\overline{W}^n_t)^2) \le 4h_n$, and $u_n^2h_n \le K$. Since $\alpha_{t-h_n}\to\alpha_{t-}$, $b_n/l_n\to 0$ and $\mathbb{P}(\mathcal{A}^n_t)\to 1$ by (SA.5), we get $\Big|u_n\widetilde{H}^n_{t,-} - \int_0^\infty\phi(s)^n_{t,-}\mathrm{d}W_s\Big| \xrightarrow{\mathbb{P}} 0$. Set $\phi(x)^n_u := \sum_{(t,\varsigma)}x_{t,\varsigma}\phi(u)^n_{t,\varsigma}$ and $\mathcal{L}^n_s := \int_0^s\phi(x)^n_u\mathrm{d}W_u$. Then $x^*\mathbf{H}_n - \mathcal{L}^n_\infty \xrightarrow{\mathbb{P}} 0$, and it remains to show that $\mathbb{E}(Ve^{i\mathcal{L}^n_\infty}) \to \mathbb{E}(V)e^{-\sigma_x^2/2}$, where $\sigma_x^2 := x^*\Sigma_H x$.

Next we show that $\langle\mathcal{L}^n,\mathcal{L}^n\rangle_\infty = \int_0^\infty(\phi(x)^n_u)^2\mathrm{d}u \xrightarrow{\mathbb{P}} \sigma_x^2$. Take n large enough that $h_n < g_0/3$, and let $\Omega^*_n := \bigcap_{t\in\mathcal{T}}(\mathcal{A}^n_t\cap\Omega(l_n)^n_t)$; then $\mathbb{P}(\Omega^*_n)\to 1$ by (SA.5) and (SA.12). On Ω^*_n, $\phi(u)^n_{t,+}$ vanishes when u is outside (t,τ_t), and $\phi(u)^n_{t,-}$ vanishes when u is outside $(t-h_n,t]$. These $2|\mathcal{T}|$ intervals are disjoint, so $\langle\mathcal{L}^n,\mathcal{L}^n\rangle_\infty = \sum_{(t,\varsigma)}x^2_{t,\varsigma}\int_0^\infty(\phi(s)^n_{t,\varsigma})^2\mathrm{d}s$.

Fix (t,ς), let $\tilde{\alpha}_{t,+} := \alpha_t$ and $\tilde{\alpha}_{t,-} := \alpha_{t-h_n}$, and for $1\le m\le l_n-1$ let $i(m)_{t,+} := n_t + l_n + 1 - m$

and $i(m)_{t,-} := a_t^n - l_n + m$. The integrand is constant on each observation interval, and counting the indicators gives, on $\mathcal{A}_t^n$, $\int_0^\infty (\phi(s)_{t,\varsigma}^n)^2 \mathrm{d}s = u_n^2 \tilde{\alpha}_{t,\varsigma} \mathfrak{E}(1)_{t,\varsigma}^n + \frac{c_n}{1+c_n} \mathfrak{E}(2)_{t,\varsigma}^n$, where $\mathfrak{E}(1)_{t,+}^n := t_{n_t+1}^n - t$, $\mathfrak{E}(1)_{t,-}^n := t - t_{a_t^n-1}^n$, and $\mathfrak{E}(2)_{t,\varsigma}^n := \frac{\tilde{\alpha}_{t,\varsigma}}{l_n^3 \Delta_n} \sum_{m=1}^{l_n-1} m^2 \, \Delta(n, i(m)_{t,\varsigma})$. By (SA.2), $\mathbb{E}(\mathfrak{E}(1)_{t,+}^n) \leq K\Delta_n$, and $\mathfrak{E}(1)_{t,-}^n \leq \sum_{j=0}^{2b_n+1} \Delta(n, a_t^n + j)$; successive conditioning and (3) give $\mathbb{E}(\mathbf{1}_{\{\mathcal{A}_t^n\}} \mathfrak{E}(1)_{t,-}^n) \leq K b_n \Delta_n$. Since $u_n^2 \Delta_n \leq l_n^{-1}$ and $b_n / l_n \to 0$, we have $u_n^2 \tilde{\alpha}_{t,\varsigma} \mathfrak{E}(1)_{t,\varsigma}^n \xrightarrow{\mathbb{P}} 0$.

Since $l_n^{-3} \sum_{m=1}^{l_n-1} m^2 \to 1/3$, it remains to show that $\mathfrak{E}(2)_{t,\varsigma}^n - l_n^{-3} \sum_{m=1}^{l_n-1} m^2 \xrightarrow{\mathbb{P}} 0$. We have, $\mathfrak{E}(2)_{t,\varsigma}^n - \frac{1}{l_n^3} \sum_{m=1}^{l_n-1} m^2 = \sum_{\ell=1}^3 \mathfrak{E}(2;\ell)_{t,\varsigma}^n$, where

$$
\begin{aligned}
\mathfrak{E}(2;1)_{t,\varsigma}^n &:= \frac{1}{l_n^3} \sum_{m=1}^{l_n-1} m^2 \Big(\frac{\tilde{\alpha}_{t,\varsigma}}{\alpha_{i(m)_{t,\varsigma}-1}^n} - 1 \Big), \\
\mathfrak{E}(2;2)_{t,\varsigma}^n &:= \frac{\tilde{\alpha}_{t,\varsigma}}{l_n^3 \Delta_n} \sum_{m=1}^{l_n-1} m^2 \Big(\mathbb{E}\big(\Delta(n, i(m)_{t,\varsigma}) \mid \mathcal{I}_{i(m)_{t,\varsigma}-1}^n\big) - \frac{\Delta_n}{\alpha_{i(m)_{t,\varsigma}-1}^n} \Big), \\
\mathfrak{E}(2;3)_{t,\varsigma}^n &:= \frac{\tilde{\alpha}_{t,\varsigma}}{l_n^3 \Delta_n} \sum_{m=1}^{l_n-1} m^2 \Big(\Delta(n, i(m)_{t,\varsigma}) - \mathbb{E}\big(\Delta(n, i(m)_{t,\varsigma}) \mid \mathcal{I}_{i(m)_{t,\varsigma}-1}^n\big) \Big).
\end{aligned}
$$

Since $l_n^{-3} \sum_{m=1}^{l_n-1} m^2 \leq 1$, we have $|\mathfrak{E}(2;1)_{t,\varsigma}^n| \leq \max_{1 \leq m \leq l_n-1} \big| \tilde{\alpha}_{t,\varsigma} / \alpha_{i(m)_{t,\varsigma}-1}^n - 1 \big| \xrightarrow{\mathbb{P}} 0$. Next, we have $|\mathfrak{E}(2;2)_{t,\varsigma}^n| \leq K\Delta_n^\eta$ by (iii) in Assumption (O). Lastly, for $\ell = 3$, we note that $\mathfrak{E}(2;3)_{t,\varsigma}^n$ is a sum of martingale differences whose conditional second moments are of order $K\Delta_n^2$. Thus, $\mathbb{E}\big((\mathfrak{E}(2;3)_{t,\varsigma}^n)^2\big) \leq K l_n^{-6} \Delta_n^{-2} \sum_{m=1}^{l_n-1} m^4 \Delta_n^2 \leq K l_n^{-1} \to 0$. Since $\mathbb{P}(\mathcal{B}_t^n) \geq \mathbb{P}(\mathcal{A}_t^n) \to 1$, we conclude that $\mathfrak{E}(2)_{t,\varsigma}^n \xrightarrow{\mathbb{P}} 1/3$, and $\langle \mathcal{L}^n, \mathcal{L}^n \rangle_\infty \xrightarrow{\mathbb{P}} \sigma_x^2$.

Finally, as in step (2) of the proof of Theorem A.4 of Jacod et al. (2017), we decompose $e^{i\mathcal{L}^n}$, after stopping, into a bounded martingale and a predictable factor of finite variation. Fix $C > \sigma_x^2$, let $\mathfrak{T}_n := \inf\{s : \langle \mathcal{L}^n, \mathcal{L}^n \rangle_s \geq C\}$ and $\widetilde{\mathcal{L}}^n := \mathcal{L}_{\cdot \wedge \mathfrak{T}_n}^n$, and let $\overline{\mathcal{L}}_s^n := \exp(i\widetilde{\mathcal{L}}_s^n + \frac{1}{2}\langle \widetilde{\mathcal{L}}^n, \widetilde{\mathcal{L}}^n \rangle_s)$. Then $\langle \widetilde{\mathcal{L}}^n, \widetilde{\mathcal{L}}^n \rangle_\infty \leq C$ and $|\overline{\mathcal{L}}^n| \leq e^{C/2}$, and Itô's formula gives $\overline{\mathcal{L}}_s^n = 1 + i \int_0^s \overline{\mathcal{L}}_u^n \phi(x)_u^n \mathbf{1}_{\{u \leq \mathfrak{T}_n\}} \mathrm{d}W_u$, so $\overline{\mathcal{L}}^n$ is a bounded $(\mathcal{F}_s^n)$-martingale. Moreover, $\mathbb{P}(\widetilde{\mathcal{L}}_\infty^n \neq \mathcal{L}_\infty^n) \leq \mathbb{P}(\langle \mathcal{L}^n, \mathcal{L}^n \rangle_\infty \geq C) \to 0$, hence $\langle \widetilde{\mathcal{L}}^n, \widetilde{\mathcal{L}}^n \rangle_\infty \xrightarrow{\mathbb{P}} \sigma_x^2$. Let $\mathcal{V}_s := \mathbb{E}(V \mid \mathcal{F}_s)$, so that $\mathcal{V}_\infty = V$. By Theorem III.4.11 of Jacod and Shiryaev (2003), $\mathcal{V} = \mathcal{V}_0 + \int_0^\cdot \mu_u \mathrm{d}W_u + N$, where μ is $(\mathcal{F}_s)$-predictable with $\mathbb{E}(\int_0^\infty \mu_u^2 \mathrm{d}u) \leq \mathbb{E}(V^2)$ and N is a square-integrable $(\mathcal{F}_s)$-martingale orthogonal to W. By Lemma A.1 of Jacod et al. (2017), $\mathcal{V}$ is an $(\mathcal{F}_s^n)$-martingale and NW is an $(\mathcal{F}_s^n)$-local martingale, so $\langle \mathcal{V}, \overline{\mathcal{L}}^n \rangle = i \int_0^\cdot \overline{\mathcal{L}}_u^n \phi(x)_u^n \mathbf{1}_{\{u \leq \mathfrak{T}_n\}} \mu_u \mathrm{d}u$ relative to $(\mathcal{F}_s^n)$. Since $\mathcal{V}$ and $\overline{\mathcal{L}}^n$ are bounded $(\mathcal{F}_s^n)$-martingales with $\overline{\mathcal{L}}_0^n = 1$, we get $\mathbb{E}(V\overline{\mathcal{L}}_\infty^n) - \mathbb{E}(V) = \mathbb{E}(\langle \mathcal{V}, \overline{\mathcal{L}}^n \rangle_\infty)$. The integrand $\phi(x)_u^n$ vanishes when u is outside the random set $\mathbb{S}_n := \bigcup_{t \in \mathcal{T}} (t - h_n, t_{n_t+l_n}^n]$, and (SA.2) gives $\mathbb{E}(\int_0^\infty \mathbf{1}_{\{\mathbb{S}_n\}}(u) \mathrm{d}u) \leq K l_n \Delta_n \to 0$. By the CS inequality,

$$
\big| \mathbb{E}(V\overline{\mathcal{L}}_\infty^n) - \mathbb{E}(V) \big| \leq e^{C/2} \sqrt{C} \left(\mathbb{E} \int_0^\infty \mathbf{1}_{\{\mathbb{S}_n\}}(u) \, \mu_u^2 \mathrm{d}u \right)^{1/2} \longrightarrow 0.
$$

Since $V e^{i\widetilde{\mathcal{L}}_\infty^n} = V \overline{\mathcal{L}}_\infty^n e^{-\langle \widetilde{\mathcal{L}}^n, \widetilde{\mathcal{L}}^n \rangle_\infty / 2}$ and $|e^{-a/2} - e^{-b/2}| \leq |a - b| \wedge 1$ for $a, b \geq 0$,

$$
\big| \mathbb{E}(V e^{i\widetilde{\mathcal{L}}_\infty^n}) - e^{-\sigma_x^2/2} \, \mathbb{E}(V) \big| \leq K e^{C/2} \, \mathbb{E}\big(|\langle \widetilde{\mathcal{L}}^n, \widetilde{\mathcal{L}}^n \rangle_\infty - \sigma_x^2| \wedge 1 \big) + \big| \mathbb{E}(V\overline{\mathcal{L}}_\infty^n) - \mathbb{E}(V) \big| \longrightarrow 0.
$$

As $\mathbb{P}(\widetilde{\mathcal{L}}^n_\infty \neq \mathcal{L}^n_\infty) \to 0$, we obtain $\mathbb{E}(Ve^{i\mathcal{L}^n_\infty}) \to \mathbb{E}(V)e^{-\sigma_x^2/2}$, which completes the proof. □

In the remaining proofs, we use the following continuations of Γ without scheduled jumps near $t \in \mathcal{T}$:

$$\check{\Gamma}(s)_{t,-} := \Gamma_{s\wedge\tau_t} - \mathbf{1}_{\{s\geq t\}}\Delta\Gamma_t \quad (s \geq 0), \qquad \check{\Gamma}(s)_{t,+} := \Gamma_{s\wedge\tau_t} \quad (s \geq t). \tag{SA.18}$$

They have the increments of $\Gamma'_{\cdot\wedge\tau_t}$ on $[\underline{\tau}_t, \infty)$ and $[t, \infty)$, respectively, and $\check{\Gamma}(s)_{t,-} = \Gamma_s$ for $s < t$ and $\check{\Gamma}(s)_{t,+} = \Gamma_s$ for $t \leq s \leq \tau_t$. Choose a bounded smooth function $f_0 : \mathbb{R} \to \mathbb{R}$ with bounded derivatives and $f_0(x) = \sqrt{x}$ on $[\underline{\Gamma}, \overline{\Gamma}]$, and set $\check{\gamma}(s)_{t,\varsigma} := f_0(\check{\Gamma}(s)_{t,\varsigma})$. Itô's formula gives the usual regular increment bounds for $\check{\gamma}$, its square, and its product with X.

Proof of Lemma SA.4. For $p \in \{0, \ldots, p_n - 1\}$, we have the following decomposition (recall $k_{n,p} = k_n + p$): $f(Y; k_{n,p})^n_i - \Gamma^n_i f(\chi; k_{n,p})^n_i = \sum_{\ell=1}^8 \mathfrak{F}(\ell)^n_i$, where

$$\begin{aligned}
&\mathfrak{F}(1)^n_i := f(X; k_{n,p})^n_i, \ \mathfrak{F}(2)^n_i := \gamma^n_i f(X, \chi; k_{n,p})^n_i,\\
&\mathfrak{F}(3)^n_i := \chi_{i+k_{n,p}} f(X, \gamma; k_{n,p})^n_i, \ \mathfrak{F}(4)^n_i := \gamma^n_i f(\chi, X; k_{n,p})^n_i,\\
&\mathfrak{F}(5)^n_i := \gamma^n_i \chi_{i+k_{n,p}} f(\chi, \gamma; k_{n,p})^n_i, \ \mathfrak{F}(6)^n_i := \chi_{i-k_{n,p}} f(\gamma, X; k_{n,p})^n_i,\\
&\mathfrak{F}(7)^n_i := \gamma^n_i \chi_{i-k_{n,p}} f(\gamma, \chi; k_{n,p})^n_i, \ \mathfrak{F}(8)^n_i := \chi_{i-k_{n,p}}\chi_{i+k_{n,p}} f(\gamma; k_{n,p})^n_i.
\end{aligned}$$

Let $\check{\mathfrak{F}}(\ell)^n_{i,\varsigma}$ be defined by replacing every occurrence of γ_s in $\mathfrak{F}(\ell)^n_i$ with $\check{\gamma}(s)_{t,\varsigma}$ defined after (SA.18). For the post-t part, write $i = n_t + m$ for deterministic $m \in \{\overline{k}_n + 1, \ldots, l_n - \overline{k}_n\}$, and let $\mathcal{H}^n_m := \mathcal{I}^n_{n_t+m-k_{n,p}} \otimes \mathcal{G}$; then $\check{\mathfrak{F}}(\ell)^n_{i,+}$ is always $\mathcal{H}^n_{m+2k_{n,p}}$-measurable. By successive conditioning, applying (A.6) and (A.7) from Jacod et al. (2017) to X and $\check{\gamma}(\cdot)_{t,+}$, and considering the boundedness of the fourth moments of χ, for any i in $\mathbb{I}^n_{t,+}$, we have

$$\mathbb{E}\big(\big|\mathbb{E}\big(\check{\mathfrak{F}}(\ell)^n_{i,+}|\mathcal{H}^n_m\big)\big|\big) \leq \begin{cases} Kk^{3/2}_{n,p}\Delta^{3/2}_n, & \text{if } \ell = 1,3,6,8;\\ Kk_{n,p}\Delta_n, & \text{if } \ell = 2,4,5,7.\end{cases}$$

We also have

$$\mathbb{E}\Big(\big|\check{\mathfrak{F}}(\ell)^n_{i,+}\big|^2\Big) \leq \begin{cases} Kk^2_{n,p}\Delta^2_n, & \text{if } \ell = 1,3,6,8;\\ Kk_{n,p}\Delta_n, & \text{if } \ell = 2,4,5,7.\end{cases}$$

On $\Omega(l_n)^n_t$, all observation times used lie in (t, τ_t), so $\check{\mathfrak{F}}(\ell)^n_{i,+} = \mathfrak{F}(\ell)^n_i$, and Lemma A.6 of Jacod et al. (2017), applied to $(\check{\mathfrak{F}}(\ell)^n_{n_t+m,+})_m$, gives

$$\mathbb{E}\Big(\mathbf{1}_{\{\Omega(l_n)^n_t\}}|l_n^{-1}\sum\nolimits_{i\in\mathbb{I}^n_{t,+}}\mathfrak{F}(\ell)^n_i|\Big) \leq \begin{cases} Kl_n^{-1/2}k^{3/2}_{n,p}\Delta_n(l_n^{1/2}\Delta_n^{1/2}\vee 1), & \text{if } \ell = 1,3,6,8;\\ Kl_n^{-1/2}k_{n,p}\Delta_n^{1/2}(l_n^{1/2}\Delta_n^{1/2}\vee 1), & \text{if } \ell = 2,4,5,7.\end{cases}$$

As these bounds are uniform in p, we get $u_n\mathbb{E}\Big(\mathbf{1}_{\{\Omega(l_n)^n_t\}}|\widehat{\Gamma}^n_{t,+} - \overline{\Gamma}^n_{t,+}|\Big) \leq K\overline{k}_n l_n^{-1}\sqrt{\frac{c_n}{1+c_n}}$. The post-$t$

part now readily follows since $(k_n \vee l_n)\Delta_n \to 0$, $\overline{k}_n/l_n \to 0$, and $\mathbb{P}(\Omega(l_n)_t^n) \to 1$.

Next for the pre-t part, define $\widehat{\Gamma}^{\circ n}_{t,-}$ and $\overline{\Gamma}^{\circ n}_{t,-}$ by replacing $\mathbb{I}^n_{t,-}$ with $\mathbb{I}^\circ(l_n)^n_{t,-}$. Take n large enough that $h_n < \min(t, g_0/3)$. On $\mathcal{B}^n_t$, the auxiliary endpoints are $\mathcal{I}^n_{a_n}$-measurable and lie within a deterministic $O(l_n)$ number of observations after a_n. We apply the same conditional estimates and the maximal version of Lemma A.6 to $\mathbf{1}_{\{\mathcal{B}^n_t\}}\check{\mathfrak{F}}(\ell)^n_{i,-}$, indexed forward from a_n. On $\mathcal{A}^n_t$, all times used by the auxiliary block lie in $(t-h_n, t)$, so $\check{\mathfrak{F}}(\ell)^n_{i,-} = \mathfrak{F}(\ell)^n_i$ for $\ell = 1, \dots, 8$, and we obtain $\mathbb{E}(\mathbf{1}_{\{\mathcal{A}^n_t\}}|\widehat{\Gamma}^{\circ n}_{t,-} - \overline{\Gamma}^{\circ n}_{t,-}|) \le K l_n^{-1/2}\overline{k}_n\Delta_n^{1/2}$.

On $\mathcal{A}^n_t$, since $0 \le n_t - a^n_t \le 2b_n$, the set $\mathbb{I}^n_{t,-} \triangle \mathbb{I}^\circ(l_n)^n_{t,-}$ is contained in the two index bands of length $2b_n$ starting at the left endpoint and just after the right endpoint of $\mathbb{I}^\circ(l_n)^n_{t,-}$. These bands are $\mathcal{I}^n_{a_n}$-measurable. Summing the continued-array second-moment bounds over them and using CS gives the unscaled bound $K(b_n/l_n)\sqrt{\overline{k}_n\Delta_n}$. Since $u_n\sqrt{\Delta_n} \le l_n^{-1/2}$, we have

$$\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}^n_t\}}u_n\Big|(\widehat{\Gamma}^n_{t,-} - \overline{\Gamma}^n_{t,-}) - (\widehat{\Gamma}^{\circ n}_{t,-} - \overline{\Gamma}^{\circ n}_{t,-})\Big|\Big) \le K\overline{k}_n^{1/2}b_n l_n^{-3/2} \longrightarrow 0.$$

Combining the estimates above and $\mathbb{P}(\mathcal{A}^n_t) \to 1$, we get the pre-$t$ part of the lemma. □

Proof of Lemma SA.5. We first decompose $\overline{f}(\chi)^n_i - \chi_i^2 = \mathfrak{G}(1)^n_i + \mathfrak{G}(2)^n_i + \mathfrak{G}(3)^n_i$, where

$$\mathfrak{G}(1)^n_i := -\chi_i\overline{\chi}^n_{i+k_n},\ \mathfrak{G}(2)^n_i := -\chi_i\overline{\chi}^n_{i-k_n},\ \mathfrak{G}(3)^n_i := \frac{1}{p_n}\sum\nolimits_{p=0}^{p_n-1}\chi_{i-k_n,p}\chi_{i+k_n,p},$$

with $\overline{\chi}^n_{i+k_n} := \frac{1}{p_n}\sum_{p=0}^{p_n-1}\chi_{i+k_n,p}$, $\overline{\chi}^n_{i-k_n} := \frac{1}{p_n}\sum_{p=0}^{p_n-1}\chi_{i-k_n,p}$. Then, by the CS inequality, we have $\mathbb{E}\Big(|\widetilde{\Gamma}^{\mathrm{tr},n}_{t,\varsigma} - \overline{\Gamma}^n_{t,\varsigma}|^2\Big) \le K\sum_{\ell=1}^3\mathbb{E}\Big(\big(\sum_{i\in\mathbb{I}^n_{t,\varsigma}}\Gamma^n_i\mathfrak{G}(\ell)^n_i/l_n\big)^2\Big)$.

Since χ is independent of $\mathcal{F}^{(0)}$ and Γ is bounded, for each $\ell \in \{1,2,3\}$, we have

$$\mathbb{E}\left[\left(\frac{1}{l_n}\sum\nolimits_{i\in\mathbb{I}^n_{t,\varsigma}}\Gamma^n_i\mathfrak{G}(\ell)^n_i\right)^2\,\middle|\,\mathcal{F}^{(0)}\right] \le \frac{K}{l_n^2}\sum\nolimits_{i,j\in\mathbb{I}^n_{t,\varsigma}}\left|\mathbb{E}\big(\mathfrak{G}(\ell)^n_i\mathfrak{G}(\ell)^n_j\big)\right|.$$

Write $r(h) := \mathbb{E}(\chi_0\chi_h)$ for $h \in \mathbb{Z}$, so that $|r(h)| \le K(1+|h|)^{-v}$ by (SA.1). For $\ell = 1$ and $0 \le k \le k_n/2$, the indices $i, i+k$ are separated from $i+k_n+p, i+k+k_n+p'$ by at least $k_n/2$. Hence (SA.1) yields $\Big|\mathbb{E}(\mathfrak{G}(1)^n_i\mathfrak{G}(1)^n_{i+k}) - \frac{r(k)}{p_n^2}\sum_{p,p'=0}^{p_n-1}r(k+p'-p)\Big| \le Kk_n^{-v}$, and, since $\sum_{h\in\mathbb{Z}}|r(h)| < \infty$, the centering term is at most $Kp_n^{-1}(1+k)^{-v}$. If $k > k_n/2$, the index i is separated from the other three indices by more than $k_n/2$, and the centering of χ_i gives the bound Kk_n^{-v}. These estimates lead to $\left|\mathbb{E}(\mathfrak{G}(1)^n_i\mathfrak{G}(1)^n_{i+k})\right| \le K\left(p_n^{-1}(1+k)^{-v} + k_n^{-v}\right),\ k \ge 0$. The same estimate holds for $\ell = 2$. For $\ell = 3$, put $q = k_n + p$ and $q' = k_n + p'$, with $0 \le p, p' < p_n$. When $0 \le k \le k_n$, the left pair $i-q, i+k-q'$ and the right pair $i+q, i+k+q'$ are separated by at least k_n. Hence $\Big|\mathbb{E}(\mathfrak{G}(3)^n_i\mathfrak{G}(3)^n_{i+k}) - \frac{1}{p_n^2}\sum_{p,p'=0}^{p_n-1}r(k+p-p')r(k-p+p')\Big| \le Kk_n^{-v}$. The centering term is bounded by $\frac{K}{p_n}\sum_{m\in\mathbb{Z}}(1+|k-m|)^{-v}(1+|k+m|)^{-v} \le \frac{K}{p_n}(1+k)^{-v}$. If $k > k_n$, one of the two exterior gaps among the four sorted indices is at least k_n: for disjoint intervals $[i-q, i+q]$ and $[i+k-q', i+k+q']$ an exterior gap is at least $2k_n$; for nested intervals one is at least $2k$; and otherwise the two exterior gaps sum to $2k$. Isolating the corresponding centered endpoint yields Kk_n^{-v}.

Thus, the same bound holds for all three components and all $k \geq 0$. Summing it over $i, j \in \mathbb{I}^n_{t,\varsigma}$ and using $v > 1$, we obtain $\mathbb{E}\left[\left(\frac{1}{l_n}\sum_{i\in\mathbb{I}^n_{t,\varsigma}} \Gamma^n_i \mathfrak{G}(\ell)^n_i\right)^2\right] \leq K\left(\frac{1}{p_n l_n} \vee k_n^{-v}\right)$, which proves the lemma. □

Proof of Lemma SA.6. Let $\mathfrak{H}^n_{t,\varsigma} := \widetilde{\Gamma}^n_{t,\varsigma} - \widetilde{\Gamma}^{\mathrm{tr},n}_{t,\varsigma} = \frac{1}{l_n}\sum_{i\in\widehat{\mathbb{I}}^n_{t,\varsigma}\setminus\mathbb{I}^n_{t,\varsigma}} (\varepsilon^n_i)^2$. Since $\mathbb{E}(\chi_i^2) = 1$ and χ is independent of $\mathcal{F}^{(0)}$, by the boundedness of Γ and $|\widehat{\mathbb{I}}^n_{t,\varsigma} \setminus \mathbb{I}^n_{t,\varsigma}| = 2\overline{k}_n$, we get $\mathbb{E}\big(|\mathfrak{H}^n_{t,\varsigma}|\big) \leq K\overline{k}_n/l_n$ for both $\varsigma \in \{+,-\}$. If $u_n\overline{k}_n/l_n \to 0$, Markov's inequality then gives $u_n\mathfrak{H}^n_{t,\varsigma} \xrightarrow{\mathbb{P}} 0$ for any $\varsigma \in \{+,-\}$. The lemma is now proved. □

We now state auxiliary lemmas for the consistency of the asymptotic variance estimator. Recall that $q_n = \lfloor L_n/d_n \rfloor$, and write $\delta_n := d_n^2\Delta_n$. For $q \in \{0,\dots,q_n-1\}$, we let $I^n_{t,+} := n_t$ and $I^n_{t,-} := n_t - L_n$, and define $\mathbb{I}(q)^n_{t,\varsigma} := \{I^n_{t,\varsigma} + qd_n + 2\overline{k}_n, \dots, I^n_{t,\varsigma} + (q+1)d_n - 2\overline{k}_n\}$, $m(q)^n_{t,\varsigma} := I^n_{t,\varsigma} + qd_n$, $T(q)^n_{t,\varsigma} := t^n_{m(q)^n_{t,\varsigma}}$, and $\mathcal{H}^n_{q,\varsigma} := \mathcal{F}^{(0)} \otimes \mathcal{G}_{m(q-1)^n_{t,\varsigma}}$. Define $\widetilde{\Gamma}(q)^n_{t,\varsigma} := \frac{1}{d_n}\sum_{i\in\mathbb{I}(q)^n_{t,\varsigma}} \Gamma^n_i\chi_i^2$, $\widetilde{\Theta}^n_{q,\varsigma} := \widetilde{\Gamma}(q)^n_{t,\varsigma} - \widetilde{\Gamma}(q-1)^n_{t,\varsigma}$ for $1 \leq q < q_n$, and the oracle estimator of the asymptotic variance $\widetilde{\Sigma}^n_{t,\varsigma} \equiv \widetilde{\Sigma}(L_n,d_n)^n_{t,\varsigma} := \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}(\widetilde{\Theta}^n_{q,\varsigma})^2$.

For the pre-t part, we extend the construction before Lemma SA.2 to blocks. Let $j_n = L_n$, take $h_n, a_n, J^n_t, \widetilde{J}^n_t, b_n$ and $\mathcal{A}^n_t$ as in Lemma SA.2, and let $I(s)^n_t := a_n + \widetilde{J}^n_t + s - j_n$ for integer shifts s, $\mathcal{S}_n := \{-b_n,\dots,b_n\}$, $r_n := \lfloor b_n/d_n \rfloor + 1$ and $\mathcal{Q}_n := \{1-r_n,\dots,q_n-1+r_n\}$. Define the family of auxiliary index sets by

$$\begin{aligned} &\mathbb{I}(q;s)^n_{t,-} := \{I(s)^n_t + qd_n + 2\overline{k}_n, \dots, I(s)^n_t + (q+1)d_n - 2\overline{k}_n\}, \\ &a(s)^n_q := I(s)^n_t + (q-1)d_n + 2\overline{k}_n. \end{aligned} \tag{SA.19}$$

We use four properties of this family. First, $I(s)^n_t = n_t - j_n = I^n_{t,-}$ when $s = s^*_n := J^n_t - \widetilde{J}^n_t$. Thus, on $\mathcal{A}^n_t$, we have $s^*_n \in \mathcal{S}_n$ and $\mathbb{I}(q;s^*_n)^n_{t,-} = \mathbb{I}(q)^n_{t,-}$ for $0 \leq q < q_n$. Second, writing $s = m(s)d_n + p(s)$ with $0 \leq p(s) < d_n$, we have $\mathbb{I}(q;s)^n_{t,-} = \mathbb{I}(q+m(s);p(s))^n_{t,-}$ and $|m(s)| \leq r_n$ for $s \in \mathcal{S}_n$. Thus, $\mathcal{Q}_n$ contains every block index $q + m(s)$ with $1 \leq q < q_n$. Every array $V(s)_q$ below depends only on $\mathbb{I}(q;s)^n_{t,-}$ and $\mathbb{I}(q-1;s)^n_{t,-}$, so it has the translation property $V(s)_q = V(p(s))_{q+m(s)}$ for $s \in \mathcal{S}_n$ and $1 \leq q < q_n$. Third, if $j_n \to \infty$ and $d_n/j_n \to 0$, then, for all large n, every index $I(s)^n_t + k$ with $s \in \mathcal{S}_n \cup \{0,\dots,d_n-1\}$ and deterministic $k \geq -(r_n+2)d_n$ equals $a_n + \ell$ for some $\mathcal{I}^n_{a_n}$-measurable $\ell > 0$, since $\widetilde{J}^n_t$ is $\mathcal{F}_{t-h_n}$-measurable and $\widetilde{J}^n_t \geq 2j_n - 2$. Hence the observation times at the indices in $\mathbb{I}(q;s)^n_{t,-}$, $q \geq -r_n$, at their shifts by at most $2d_n$, and at $a(s)^n_q$ are stopping times, and $(\mathcal{I}^n_{a(s)^n_q})_{q\in\mathcal{Q}_n}$ is a filtration (the stopping-time property of the family).

Fourth, the family admits a maximal bound over phases. Let $\kappa \geq 1$, $\varpi_n > 0$, and let $V(0),\dots,V(d_n-1)$ be random variables with $\mathbb{E}(|V(p)|^\kappa) \leq K\varpi_n$ for $0 \leq p < d_n$ and $\mathbb{E}(|V(p+1) - V(p)|^\kappa) \leq K\varpi_n d_n^{-\kappa}$ for $0 \leq p < d_n - 1$. For $z_0,\dots,z_{d-1} \in \mathbb{R}$, the triangle inequality and Hölder's inequality give $\max_{0\leq i\leq d-1}|z_i|^\kappa \leq \frac{K}{d}\sum_{i=0}^{d-1}|z_i|^\kappa + Kd^{\kappa-1}\sum_{i=0}^{d-2}|z_{i+1} - z_i|^\kappa$. Taking

$z_p = V(p)$ and $d = d_n$, and then expectations, we obtain

$$\mathbb{E}\Big(\max_{0\le p\le d_n-1} |V(p)|^\kappa\Big) \le K\varpi_n. \tag{SA.20}$$

Now let $(V(s)_q)$ be an array with the translation property such that, for each $q \in \mathcal{Q}_n$, the variables $(V(p)_q)_{0\le p<d_n}$ satisfy these bounds with K not depending on q. On $\mathcal{A}_t^n$, we have $|m(s_n^*)| \le r_n$, so the translation property gives $\sum_{q=1}^{q_n-1} |V(s_n^*)_q|^\kappa \le \sum_{q\in\mathcal{Q}_n} \max_{0\le p<d_n} |V(p)_q|^\kappa$. Hence, by (SA.20) and $|\mathcal{Q}_n| \le Kq_n$,

$$\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}_t^n\}} \sum\nolimits_{q=1}^{q_n-1} |V(s_n^*)_q|^\kappa\Big) \le Kq_n\varpi_n. \tag{SA.21}$$

Next we introduce a useful decomposition of $\widetilde{\Gamma}(q)_{t,\varsigma}^n$ and a triangular summation identity. With $\eta_i := (\chi_i)^2 - 1$, we decompose $\widetilde{\Gamma}(q)_{t,\varsigma}^n = U_{q,\varsigma}^n + R_{q,\varsigma}^n$, where

$$U_{q,\varsigma}^n := \frac{1}{d_n} \sum_{i\in\mathbb{I}(q)_{t,\varsigma}^n} \Gamma_i^n \eta_i,\ R_{q,\varsigma}^n := \frac{1}{d_n} \sum_{i\in\mathbb{I}(q)_{t,\varsigma}^n} \Gamma_i^n,\ \Delta U_{q,\varsigma}^n := U_{q,\varsigma}^n - U_{q-1,\varsigma}^n,\ \Delta R_{q,\varsigma}^n := R_{q,\varsigma}^n - R_{q-1,\varsigma}^n.$$

For the pre-t case, we additionally define $\check{R}(s)_{q,-}^n := \frac{1}{d_n}\sum_{i\in\mathbb{I}(q;s)_{t,-}^n} \check{\Gamma}(t_i^n)_{t,-}$. We let $\Omega_t'^n := \Omega(r_n')_t^n$ where $r_n' := L_n + 2d_n + 2b_n + 4\overline{k}_n$, which satisfies $r_n'\Delta_n \to 0$ when $L_n\Delta_n \to 0$. Let $a_{q,\varsigma}^n := I_{t,\varsigma}^n + (q-1)d_n + 2\overline{k}_n$ and $M_n := d_n - 4\overline{k}_n + 1$. For any array (V_i^n), summation by parts gives

$$\sum\nolimits_{i\in\mathbb{I}(q)_{t,\varsigma}^n} V_i^n - \sum\nolimits_{i\in\mathbb{I}(q-1)_{t,\varsigma}^n} V_i^n = \sum\nolimits_{k=1}^{2d_n-4\overline{k}_n} w_k^n (V_{a_{q,\varsigma}^n+k}^n - V_{a_{q,\varsigma}^n+k-1}^n), \tag{SA.22}$$

where $w_k^n := k$ for $1 \le k \le M_n$, $w_k^n := M_n$ for $M_n < k \le d_n$, and $w_k^n := d_n + M_n - k$ for $d_n < k \le 2d_n - 4\overline{k}_n$. The identity also holds for the auxiliary blocks $\mathbb{I}(q;s)_{t,-}^n$, with $a(s)_q^n$ in place of $a_{q,-}^n$.

The following lemma collects several estimates for moments of quantities related to $U_{q,\varsigma}^n$ and $R_{q,\varsigma}^n$.

Lemma SA.11. *Let Assumption (S-HONV) hold with $v > 2$, and assume that $d_n, L_n \to \infty$, $\overline{k}_n/d_n \to 0$, $d_n/L_n \to 0$, and $L_n\Delta_n \to 0$. For $\varsigma \in \{+,-\}$ and $q \in \{1,\dots,q_n-1\}$, we have*

$$\mathbb{E}\Big((\Delta U_{q,\varsigma}^n)^2 | \mathcal{F}^{(0)}\Big) \le Kd_n^{-1}, \quad \mathbb{E}\Big((\Delta U_{q,\varsigma}^n)^4 | \mathcal{F}^{(0)}\Big) \le Kd_n^{-2}; \tag{SA.23}$$

and, in the post-t and pre-t cases, respectively,

$$\mathbb{E}\Big(\mathbf{1}_{\{\Omega_t'^n\}} (\Delta R_{q,+}^n)^2\Big) \le Kd_n\Delta_n, \quad \mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}_t^n\}} \sum\nolimits_{q=1}^{q_n-1} (\Delta R_{q,-}^n)^2\Big) \le KL_n\Delta_n. \tag{SA.24}$$

Proof. By (SA.1), $|\mathbb{E}(\eta_\ell\eta_{\ell'})| \le K(1+|\ell-\ell'|)^{-v}$. Using this, the boundedness of Γ, and the independence of χ and $\mathcal{F}^{(0)}$, as $v > 2$, for any $a \in \{q-1, q\}$ and $\varsigma \in \{+,-\}$, we have $\mathbb{E}((U_{a,\varsigma}^n)^2|\mathcal{F}^{(0)}) \le Kd_n^{-2}\sum_{\ell,\ell'}(1+|\ell-\ell'|)^{-v} \le Kd_n^{-1}$. By adding the two blocks, the first bound in

(SA.23) follows. Then, by applying Lemma SA.7 with $p=4$ conditional on $\mathcal{F}^{(0)}$, and the boundedness of Γ, we have $\mathbb{E}((U^n_{a,\varsigma})^4|\mathcal{F}^{(0)}) \le Kd_n^{-2}$. So, we obtain the second part of (SA.23) by adding the two blocks.

On Ω'^n_t, we can replace Γ in $\Delta R^n_{q,+}$ with $\check{\Gamma}(\cdot)_{t,+}$, whose increments after t are those of the regular component stopped at τ_t. Applying (SA.22) to $\Delta R^n_{q,+}$ and using the martingale property and $\sum_k (w^n_k)^2 \le Kd_n^3$, we get $\mathbb{E}(\mathbf{1}_{\{\Omega'^n_t\}}(\Delta R^n_{q,+})^2) \le Kd_n^{-2}\sum_k (w^n_k)^2\Delta_n \le Kd_n\Delta_n$. The first bound in (SA.24) follows. Then, by the stopping-time property after (SA.19), applying the same arguments to $\Delta\check{R}(s)^n_{q,-}$, we obtain $\mathbb{E}\big((\Delta\check{R}(s)^n_{q,-})^2\big) \le Kd_n\Delta_n$ for $0 \le s < d_n$ and $q \in \mathcal{Q}_n$.

We rewrite $\Delta\check{R}(s+1)^n_{q,-} - \Delta\check{R}(s)^n_{q,-} = d_n^{-1}\sum_k \Delta w^n_k\{\check{\Gamma}(t^n_{a(s)^n_q+k})_{t,-} - \check{\Gamma}(t^n_{a(s)^n_q+k-1})_{t,-}\}$ for any fixed t and q, where $\Delta w^n_k := w^n_{k-1} - w^n_k$. Therefore, we have

$$\mathbb{E}\big((\Delta\check{R}(s+1)^n_{q,-} - \Delta\check{R}(s)^n_{q,-})^2\big) \le K(d_n^{-2}\sum\nolimits_k (\Delta w^n_k)^2\Delta_n + (d_n^{-1}\sum\nolimits_k |\Delta w^n_k|\Delta_n)^2) \le Kd_n^{-1}\Delta_n.$$

Thus, by (SA.21) with $\kappa=2$ and $\varpi_n = d_n\Delta_n$, and since $\Delta\check{R}(s^*_n)^n_{q,-} = \Delta R^n_{q,-}$ on $\mathcal{A}^n_t$, we obtain the second bound in (SA.24). The proof is now complete. □

Lemma SA.12. *Let Assumption (S-HONV) hold, and $d_n \wedge q_n \to \infty$, $L_n\Delta_n \to 0$, $k_n \wedge p_n \to \infty$, $\overline{k}_n/d_n \to 0$, and $v > 2$. If $k_n^{-v}\left(d_n \wedge \frac{1}{d_n\Delta_n}\right) \to 0$ and $\overline{k}_n\left(\frac{1}{d_n} \vee d_n\Delta_n\right) \to 0$, for $\varsigma \in \{+,-\}$, we have $\frac{1}{1+\delta_n}(\Sigma(L_n,d_n)^n_{t,\varsigma} - \widetilde{\Sigma}(L_n,d_n)^n_{t,\varsigma}) \xrightarrow{\mathbb{P}} 0$.*

Proof. We first consider some useful estimates. For any (L_n,d_n), $t \in \mathcal{T}$ and $\varsigma \in \{+,-\}$, let $\mathfrak{H}(1)^n_{q,\varsigma} = \widehat{\Gamma}(q)^n_{t,\varsigma} - \overline{\Gamma}(q)^n_{t,\varsigma}$, $\mathfrak{H}(2)^n_{q,\varsigma} = \overline{\Gamma}(q)^n_{t,\varsigma} - \widetilde{\Gamma}(q)^n_{t,\varsigma}$, where $\overline{\Gamma}^n_{t,\varsigma}$ is defined in Lemma SA.4 (defined on $\mathbb{I}(q)^n_{t,\varsigma}$ now). We further denote $\Delta\mathfrak{H}^n_{q,\varsigma} := \Delta\mathfrak{H}(1)^n_{q,\varsigma} + \Delta\mathfrak{H}(2)^n_{q,\varsigma}$ where $\Delta\mathfrak{H}(\ell)^n_{q,\varsigma} := \mathfrak{H}(\ell)^n_{q,\varsigma} - \mathfrak{H}(\ell)^n_{q-1,\varsigma}$ for $\ell = 1,2$. Let $\epsilon_n := p_n^{-1} \vee d_nk_n^{-v} \vee d_n\overline{k}_n\Delta_n$.

We decompose $\Sigma(L_n,d_n)^n_{t,\varsigma} - \widetilde{\Sigma}(L_n,d_n)^n_{t,\varsigma} = \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}((\Delta\mathfrak{H}^n_{q,\varsigma})^2 + 2\Delta\mathfrak{H}^n_{q,\varsigma}\widetilde{\Theta}^n_{q,\varsigma})$. For the post-$t$ case, by Lemma SA.4, we have $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}(\mathfrak{F}(\ell)^n_i)^2\Big) \le K\overline{k}_n\Delta_n$ for any $i \in \mathbb{I}^n_{t,+}$ and $\ell \in \{1,\dots,8\}$. Thus, we get $\mathbb{E}(\mathbf{1}_{\{\Omega'^n_t\}}(\mathfrak{H}(1)^n_{q,+})^2) \le K\overline{k}_n\Delta_n$ by the CS inequality. By the estimate above, Lemma SA.5, and the CS inequality, for $q,q' \in \{0,\dots,q_n-1\}$, we have

$$\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}\big|\mathfrak{H}(1)^n_{q,+}\mathfrak{H}(1)^n_{q',+}\big|\Big) \le K\overline{k}_n\Delta_n, \quad \mathbb{E}\big(\big|\mathfrak{H}(2)^n_{q,\varsigma}\mathfrak{H}(2)^n_{q',\varsigma}\big|\big) \le K\left(\frac{1}{p_nd_n}\bigvee\frac{1}{k_n^v}\right),$$
$$\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}\big|\mathfrak{H}(1)^n_{q,+}\mathfrak{H}(2)^n_{q',+}\big|\Big) \le K(\overline{k}_n\Delta_n)^{1/2}\left(p_n^{-1/2}d_n^{-1/2} \vee k_n^{-v/2}\right).$$

Applying (SA.23) and the first bound in (SA.24), we have $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}(\widetilde{\Theta}^n_{q,+})^2\Big) \le K(d_n^{-1} \vee d_n\Delta_n)$. Thus, by the CS inequality, for $q,q' \in \{1,\dots,q_n-1\}$, we have

$$\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}\Big|\mathfrak{H}(1)^n_{q,+}\widetilde{\Theta}^n_{q',+}\Big|\Big) \le K(\overline{k}_n\Delta_n)^{1/2}(d_n^{-1/2} \vee (d_n\Delta_n)^{1/2}),$$
$$\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}\Big|\mathfrak{H}(2)^n_{q,+}\widetilde{\Theta}^n_{q',+}\Big|\Big) \le K\left(p_n^{-1/2}d_n^{-1/2} \vee k_n^{-v/2}\right)(d_n^{-1/2} \vee (d_n\Delta_n)^{1/2}).$$

Hence, we have $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}|\Sigma(L_n,d_n)^n_{t,+} - \widetilde{\Sigma}(L_n,d_n)^n_{t,+}|\Big) \le K(\epsilon_n + (\epsilon_n(1 \vee \delta_n))^{1/2})$. Therefore, the

post-t case of the lemma readily follows given the asymptotic conditions.

For the pre-t case, by (SA.23), we obtain $\mathbb{E}(\sum_{q=1}^{q_n-1}(\Delta U^n_{q,-})^2) \le K d_n^{-1} q_n$. Then, for a given $t \in \mathcal{T}$ and $q \in \{1,\dots,q_n-1\}$, we define $\check{\mathfrak{H}}(1;s)^n_{q,-}$ by replacing γ with $\check{\gamma}(\cdot)_{t,-}$ in every noise factor of $\mathfrak{H}(1)^n_{q,-}$, and each variance weight Γ^n_i in $\overline{\Gamma}(q)^n_{t,-}$ with $\check{\gamma}(t^n_i)^2_{t,-}$, and replacing $\mathbb{I}(q)^n_{t,-}$ with $\mathbb{I}(q;s)^n_{t,-}$. Thus, by the estimates for $\mathfrak{F}(\ell)^n_i$ in the proof of Lemma SA.4 for $\ell \in \{1,\dots,8\}$ and the CS inequality, we have

$$\mathbb{E}\big(|\Delta\check{\mathfrak{H}}(1;s)^n_{q,-}|^2\big) \le K\overline{k}_n\Delta_n, \quad \mathbb{E}\big(|\Delta\check{\mathfrak{H}}(1;s+1)^n_{q,-} - \Delta\check{\mathfrak{H}}(1;s)^n_{q,-}|^2\big) \le K d_n^{-2}\overline{k}_n\Delta_n.$$

Therefore, we have $\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}(\Delta\check{\mathfrak{H}}(1;s^*_n)^n_{q,-})^2\Big) \le K q_n\overline{k}_n\Delta_n$ by (SA.21) with $\kappa = 2$ and $\varpi_n = \overline{k}_n\Delta_n$. At $s = s^*_n$, the trimmed blocks and all the lagged observations used in $\mathfrak{H}(1)$ are strictly before t. Hence $\Delta\mathfrak{H}(1)^n_{q,-} = \Delta\check{\mathfrak{H}}(1;s^*_n)^n_{q,-}$ on $\mathcal{A}^n_t$. So, we obtain $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\cap\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}(\Delta\mathfrak{H}(1)^n_{q,-})^2\Big) \le K q_n\overline{k}_n\Delta_n$. Then, combining with Lemma SA.5, we get $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\cap\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}(\Delta\mathfrak{H}^n_{q,-})^2\Big) \le K d_n^{-1} q_n\epsilon_n$.

Using the second bound in (SA.24), we have $\mathbb{E}(\mathbf{1}_{\{\Omega'^n_t\cap\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}(\Delta R^n_{q,-})^2) \le K L_n\Delta_n$. Then, applying the CS inequality to these block sums gives the following estimates for the two cross terms: $\frac{d_n}{2q_n}\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\cap\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}|\Delta\mathfrak{H}^n_{q,-}\Delta U^n_{q,-}|\Big) \le K\epsilon_n^{1/2}$ and $\frac{d_n}{2q_n}\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\cap\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}|\Delta\mathfrak{H}^n_{q,-}\Delta R^n_{q,-}|\Big) \le K(\epsilon_n\delta_n)^{1/2}$. Combining these estimates gives the bound $\mathbb{E}(\mathbf{1}_{\{\Omega'^n_t\cap\mathcal{A}^n_t\}}|\Sigma(L_n,d_n)^n_{t,-} - \widetilde{\Sigma}(L_n,d_n)^n_{t,-}|) \le K\{\epsilon_n + (\epsilon_n(1\vee\delta_n))^{1/2}\}$. The assumed rates give $\epsilon_n/(1+\delta_n) \to 0$. Together with $\mathbb{P}(\Omega'^n_t) \to 1$ and $\mathbb{P}(\mathcal{A}^n_t) \to 1$, we obtain the pre-$t$ case of the lemma by Markov's inequality. The proof is now complete. □

We now make the following decomposition $\widetilde{\Sigma}^n_{t,\varsigma} = \widetilde{\Sigma}^{U,n}_{t,\varsigma} + \widetilde{\Sigma}^{R,n}_{t,\varsigma} + 2\widetilde{\Sigma}^{UR,n}_{t,\varsigma}$, where

$$\widetilde{\Sigma}^{U,n}_{t,\varsigma} := \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}(\Delta U^n_{q,\varsigma})^2, \quad \widetilde{\Sigma}^{R,n}_{t,\varsigma} := \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}(\Delta R^n_{q,\varsigma})^2, \quad \widetilde{\Sigma}^{UR,n}_{t,\varsigma} := \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}\Delta U^n_{q,\varsigma}\Delta R^n_{q,\varsigma}.$$

Let $\mathcal{E}^n_{t,+} := \Omega'^n_t$ and $\mathcal{E}^n_{t,-} := \Omega'^n_t \cap \mathcal{A}^n_t$. We first introduce the following lemma giving the estimate for $\widetilde{\Sigma}^{U,n}_{t,\varsigma}$.

Lemma SA.13. *Let Assumption (S-HONV) hold. Assume that $d_n, q_n, L_n \to \infty$, $L_n\Delta_n \to 0$, $k_n \wedge p_n \to \infty$, $(k_n \vee p_n)/d_n \to 0$, and $v > 2$. Then, for $\varsigma \in \{+,-\}$, we have*

$$\frac{1}{q_n}\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{E}^n_{t,\varsigma}\}}\sum_{q=1}^{q_n-1}|d_n\mathbb{E}\big((\Delta U^n_{q,\varsigma})^2|\mathcal{H}^n_{q,\varsigma}\big) - 2\Gamma^2_{T(q)^n_{t,\varsigma}}\mathfrak{S}|\Big) \le K((d_n\Delta_n)^{1/2} + \overline{k}_n d_n^{-1}). \tag{SA.25}$$

Proof. For $q \in \{1,\dots,q_n-1\}$ and $a,b \in \{q-1,q\}$, let $\mathfrak{I}(a,b)^n_{q,\varsigma} := d_n\mathbb{E}\Big(U^n_{a,\varsigma}U^n_{b,\varsigma}|\mathcal{H}^n_{q,\varsigma}\Big)$ and $c(k) := \mathbb{E}\big((\chi^2_0-1)(\chi^2_k-1)\big)$, so $\mathfrak{S} = \sum_{k=-\infty}^{\infty}c(k)$. We denote $m(q-1)^n_{t,\varsigma}$ as $m^n_{q,t,\varsigma}$ for short. For $\ell,\ell' \ge m^n_{q,t,\varsigma} + \overline{k}_n$ where $q \in \{1,\dots,q_n-1\}$, we have

$$\mathbb{E}\Big(\Big|\mathbb{E}\Big(\eta_\ell\eta_{\ell'}|\mathcal{G}_{m^n_{q,t,\varsigma}}\Big) - \mathbb{E}(\eta_\ell\eta_{\ell'})\Big|\Big) \le K((1+\ell\wedge\ell'-m^n_{q,t,\varsigma})^{-v} \wedge (1+|\ell-\ell'|)^{-v}). \tag{SA.26}$$

Then, we consider the terms in the same block, that is, $a = b \in \{q-1, q\}$. We decompose $\mathfrak{I}(a,a)^n_{q,\varsigma} - \Gamma^2_{T(q)^n_{t,\varsigma}}\mathfrak{S} = \sum_{m=1}^3 \mathfrak{I}(a;m)^n_{q,\varsigma}$ where

$$\mathfrak{I}(a;1)^n_{q,\varsigma} := \frac{1}{d_n}\sum\nolimits_{\ell,\ell'\in\mathbb{I}(a)^n_{t,\varsigma}} \Gamma^n_\ell\Gamma^n_{\ell'}\left(\mathbb{E}\Big(\eta_\ell\eta_{\ell'}|\mathcal{G}_{m^n_{q,t,\varsigma}}\Big) - c(\ell'-\ell)\right),$$
$$\mathfrak{I}(a;2)^n_{q,\varsigma} := \frac{1}{d_n}\sum\nolimits_{\ell,\ell'\in\mathbb{I}(a)^n_{t,\varsigma}} (\Gamma^n_\ell\Gamma^n_{\ell'} - \Gamma^2_{T(q)^n_{t,\varsigma}})c(\ell'-\ell),$$
$$\mathfrak{I}(a;3)^n_{q,\varsigma} := \left(\frac{\Gamma^2_{T(q)^n_{t,\varsigma}}}{d_n}\sum\nolimits_{\ell,\ell'\in\mathbb{I}(a)^n_{t,\varsigma}} c(\ell'-\ell)\right) - \Gamma^2_{T(q)^n_{t,\varsigma}}\mathfrak{S}.$$

By (SA.26) and the boundedness of Γ, we have $\mathbb{E}\big(|\mathfrak{I}(a;1)^n_{q,\varsigma}|\big) \le Kd_n^{-1}\overline{k}_n^{2-v}$. By (SA.1) and $v > 2$, both $\sum_{h\in\mathbb{Z}}|c(h)|$ and $\sum_{h\in\mathbb{Z}}|h||c(h)|$ are finite. Therefore, since $|\mathbb{I}(a)^n_{t,\varsigma}| = d_n - 4\overline{k}_n + 1$, $\mathbb{E}\big(|\mathfrak{I}(a;3)^n_{q,\varsigma}|\big) \le K(\overline{k}_n/d_n + d_n^{1-v} + 1/d_n) \le K\overline{k}_n/d_n$.

Now for the post-t case, we have $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}|\Gamma^n_\ell\Gamma^n_{\ell'} - \Gamma^2_{T(q)^n_{t,+}}|\Big) \le K(d_n\Delta_n)^{1/2}$ by applying (A.6) in Jacod et al. (2017). Together with the boundedness of $\frac{1}{d_n}\sum_{\ell,\ell'\in\mathbb{I}(a)^n_{t,\varsigma}}|c(\ell'-\ell)|$, we obtain $\mathbb{E}\Big(\mathbf{1}_{\{\Omega'^n_t\}}|\mathfrak{I}(a;2)^n_{q,+}|\Big) \le K(d_n\Delta_n)^{1/2}$.

For the pre-t part, let $T(q;s)^n_{t,-} := t^n_{I(s)^n_t + qd_n}$, we define

$$\check{\mathfrak{I}}(a;2;s)^n_{q,-} := \frac{1}{d_n}\sum\nolimits_{\ell,\ell'\in\mathbb{I}(a;s)^n_{t,-}} c(\ell'-\ell)\times\big(\check{\Gamma}(t^n_\ell)_{t,-}\check{\Gamma}(t^n_{\ell'})_{t,-} - \check{\Gamma}(T(q;s)^n_{t,-})^2_{t,-}\big).$$

By the stopping-time property discussed after (SA.19), boundedness of $\check{\Gamma}(\cdot)_{t,-}$, we obtain $\mathbb{E}\Big(\max_{0\le r\le d_n-1}\sum_{a\in\{q-1,q\}}|\check{\mathfrak{I}}(a;2;r)^n_{q,-}|\Big) \le K(d_n\Delta_n)^{1/2}$. Then, by the translation property to obtain (SA.21), we get $\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}^n_t\}}\sum_{q=1}^{q_n-1}\sum_{a\in\{q-1,q\}}|\check{\mathfrak{I}}(a;2;s^*_n)^n_{q,-}|\Big) \le Kq_n(d_n\Delta_n)^{1/2}$, where s^*_n is defined after (SA.19).

Lastly, we consider the cross-block term $\mathfrak{I}(q-1,q)^n_{q,\varsigma}$. Note that there are at most h pairs $(\ell,\ell')\in\mathbb{I}(q-1)^n_{t,\varsigma}\times\mathbb{I}(q)^n_{t,\varsigma}$ satisfying $\ell'-\ell = h$. Hence, by the boundedness of Γ, $\sum_{h\in\mathbb{Z}}|h||c(h)| < \infty$, and (SA.26) as for $\mathfrak{I}(a;1)^n_{q,\varsigma}$, summing over these pairs gives

$$\mathbb{E}\big(|\mathfrak{I}(q-1,q)^n_{q,\varsigma}|\big) \le \frac{K}{d_n}\sum\nolimits_{\ell,\ell'}\left(|c(\ell'-\ell)| + \mathbb{E}\Big(\Big|\mathbb{E}\Big(\eta_\ell\eta_{\ell'}|\mathcal{G}_{m^n_{q,t,\varsigma}}\Big) - c(\ell'-\ell)\Big|\Big)\right) \le Kd_n^{-1}.$$

Finally, $d_n\mathbb{E}\big((\Delta U^n_{q,\varsigma})^2|\mathcal{H}^n_{q,\varsigma}\big) = \mathfrak{I}(q,q)^n_{q,\varsigma} + \mathfrak{I}(q-1,q-1)^n_{q,\varsigma} - 2\mathfrak{I}(q-1,q)^n_{q,\varsigma}$. Since $\mathcal{E}^n_{t,+} = \Omega'^n_t$ and $\mathcal{E}^n_{t,-}\subseteq\mathcal{A}^n_t$, summing the above bounds over q and using $d_n^{-1}\overline{k}_n^{2-v} \le d_n^{-1} \le \overline{k}_n/d_n$ for $v > 2$ and $\overline{k}_n \ge 1$ gives (SA.25). The proof is now complete. □

Using the transformation (SA.22), we can rewrite the diffusion-driven part of $\Delta R^n_{q,\varsigma}$ as $\Delta R'^n_{q,\varsigma} := \frac{1}{d_n}\sum_{k=1}^{2d_n-4\overline{k}_n} w^n_k\int_{t^n_{a^n_{q,\varsigma}+k-1}}^{t^n_{a^n_{q,\varsigma}+k}}\widetilde{\sigma}_s\mathrm{d}W_s$. For the pre-$t$ case, we define $\Delta R'(s)^n_{q,-}$ by replacing the index set of $\Delta R'^n_{q,-}$ with $\mathbb{I}(q;s)^n_{t,-}$. The next lemma shows that only the Brownian part $\Delta R'^n_{q,\varsigma}$ of $\Delta R^n_{q,\varsigma}$ matters. An analogous result for spot volatility estimation is Jacod and Protter (2012, Lemma 13.3.10), which shows that the jumps of the price process do not affect the one-sided local

estimators of the volatility.

Lemma SA.14. *Let Assumption (S-HONV) hold. For an arbitrary pair of (L_n, d_n), assume that $d_n, q_n \to \infty$, $L_n\Delta_n \to 0$, and $\overline{k}_n/d_n \to 0$. Then, for any $\varsigma \in \{+,-\}$, we have $\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}((\Delta R^n_{q,\varsigma})^2 - (\Delta R'^n_{q,\varsigma})^2) \xrightarrow{\mathbb{P}} 0$.*

Proof. We prove the negligibility of the drift and jump parts of $\Delta R^n_{q,\varsigma}$. For $\epsilon \in (0,1]$, we define $\widetilde{b}'(\epsilon)_t$, $M(\epsilon)_t$ and $J(\epsilon)_t$ analogously as in the proof of Lemma SA.8. We can apply (SA.22) to get $\mathfrak{J}^n_{q,\varsigma} := \Delta R^n_{q,\varsigma} - \Delta R'^n_{q,\varsigma} = \sum_{\ell=1}^3 \mathfrak{J}(\epsilon;\ell)^n_{q,\varsigma}$, where

$$\mathfrak{J}(\epsilon;1)^n_{q,\varsigma} := \frac{1}{d_n}\sum\nolimits_{k=1}^{2d_n-4\overline{k}_n} w^n_k \int_{t^n_{a^n_{q,\varsigma}+k-1}}^{t^n_{a^n_{q,\varsigma}+k}} \widetilde{b}'(\epsilon)_s \mathrm{d}s,\ \mathfrak{J}(\epsilon;2)^n_{q,\varsigma} := \frac{1}{d_n}\sum\nolimits_{k=1}^{2d_n-4\overline{k}_n} w^n_k \Delta M(\epsilon)^n_{a^n_{q,\varsigma}+k},$$

and $\mathfrak{J}(\epsilon;3)^n_{q,\varsigma} := \frac{1}{d_n}\sum_{k=1}^{2d_n-4\overline{k}_n} w^n_k \Delta J(\epsilon)^n_{a^n_{q,\varsigma}+k}$.

Let $\mathcal{D}^n_{t,+} := (t^n_{I^n_{t,+}+1}, t^n_{I^n_{t,+}+L_n}]$, $\mathcal{D}^n_{t,-} := (t^n_{I(-b_n)^n_t - d_n}, t^n_{I(b_n)^n_t + L_n + 2d_n}]$. By the stopping-time property after (SA.19), the endpoints of both sets are stopping times. Define the number of unscheduled large jumps within $\mathcal{D}^n_{t,\varsigma}$ as $N(\epsilon)^n_{t,\varsigma} := \int_{\mathcal{D}^n_{t,\varsigma}}\int_{\{J^{\Gamma'}(z)>\epsilon\}} \mathfrak{p}(\mathrm{d}s,\mathrm{d}z)$ and $\lambda_\epsilon := \int_{\{J^{\Gamma'}(z)>\epsilon\}}\lambda(\mathrm{d}z) < \infty$. Then, we have $\mathbb{P}(N(\epsilon)^n_{t,\varsigma} > 0) \le K\lambda_\epsilon L_n\Delta_n \to 0$ for any $\epsilon > 0$. On the event $\{N(\epsilon)^n_{t,\varsigma} = 0\}$, by definition, we have $\mathfrak{J}(\epsilon;3)^n_{q,\varsigma} = 0$ simultaneously for all blocks q covered by $\mathcal{D}^n_{t,\varsigma}$.

For the post-t case, $w^n_k \le d_n$ gives $|\mathfrak{J}(\epsilon;1)^n_{q,+}| \le K\sum_{k=1}^{2d_n-4\overline{k}_n}\int_{t^n_{a^n_{q,+}+k-1}}^{t^n_{a^n_{q,+}+k}} |\widetilde{b}'(\epsilon)_s|\mathrm{d}s$. Thus, a similar argument as in Lemma SA.8 yields $\mathbb{E}\big((\mathfrak{J}(\epsilon;1)^n_{q,+})^2\big) \le K\epsilon^{-2}d_n^2\Delta_n^2$. Next, we have $\mathbb{E}\big((\mathfrak{J}(\epsilon;2)^n_{q,+})^2\big) \le K\frac{\phi(\epsilon)}{d_n^2}\sum_{k=1}^{2d_n-4\overline{k}_n}(w^n_k)^2\Delta_n \le K\phi(\epsilon)d_n\Delta_n$ by the martingale property of $M(\epsilon)_t$. We obtain

$$\frac{1}{L_n\Delta_n}\mathbb{E}\left(\mathbf{1}_{\{\Omega'^n_t\}}\mathbf{1}_{\{N(\epsilon)^n_{t,+}=0\}}\sum_{q=1}^{q_n-1}(\mathfrak{J}^n_{q,+})^2\right) \le K(\epsilon^{-2}d_n\Delta_n + \phi(\epsilon)).$$

For the pre-t case, we define $\mathfrak{J}(\epsilon;s;\ell)^n_{q,-}$ by replacing the $a^n_{q,-}$ in $\mathfrak{J}(\epsilon;\ell)^n_{q,-}$ with $a(s)^n_q$ and stopping all the integrations or increments at τ_t, for any $s \in \mathcal{S}_n \cup \{0,\dots,d_n-1\}$ and $\ell \in \{1,2,3\}$.

Let $\check{\mathfrak{J}}(s)^n_{q,-} := \sum_{\ell=1}^3 \mathfrak{J}(\epsilon;s;\ell)^n_{q,-}$. We have for $0 \le s < d_n$ and $q \in \mathcal{Q}_n$

$$\mathbb{E}\Big(|\check{\mathfrak{J}}(s)^n_{q,-}|^2\mathbf{1}_{\{N(\epsilon)^n_{t,-}=0\}}\Big) \le Kd_n\Delta_n((\epsilon^{-2}d_n\Delta_n)\vee\phi(\epsilon)).$$

Using the weight-difference argument in the proof of Lemma SA.11, the phase increment of $\check{\mathfrak{J}}(s)^n_{q,-}\mathbf{1}_{\{N(\epsilon)^n_{t,-}=0\}}$ has second moment at most $K(\epsilon^{-2}\Delta_n^2 + \phi(\epsilon)d_n^{-1}\Delta_n)$. Thus (SA.21) applies to this array with $\kappa = 2$ and $\varpi_n = \epsilon^{-2}(d_n\Delta_n)^2 + \phi(\epsilon)d_n\Delta_n$. At $s = s^*_n$, all endpoints are before t, so $\check{\mathfrak{J}}(s^*_n)^n_{q,-} = \mathfrak{J}^n_{q,-}$ on $\mathcal{A}^n_t$. Hence we have $\frac{1}{L_n\Delta_n}\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}^n_t\}}\mathbf{1}_{\{N(\epsilon)^n_{t,-}=0\}}\sum_{q=1}^{q_n-1}(\mathfrak{J}^n_{q,-})^2\Big) \le K(\epsilon^{-2}d_n\Delta_n + \phi(\epsilon))$.

Recall that $\mathcal{E}^n_{t,+} = \Omega'^n_t$ and $\mathcal{E}^n_{t,-} = \Omega'^n_t \cap \mathcal{A}^n_t$, so the bounds for the two cases can be written together as $\frac{1}{L_n\Delta_n}\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{E}^n_{t,\varsigma}\}}\mathbf{1}_{\{N(\epsilon)^n_{t,\varsigma}=0\}}\sum_{q=1}^{q_n-1}(\mathfrak{J}^n_{q,\varsigma})^2\Big) \le K(\epsilon^{-2}d_n\Delta_n + \phi(\epsilon))$; then, by (SA.24), we also obtain $\frac{1}{L_n\Delta_n}\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{E}^n_{t,\varsigma}\}}\sum_{q=1}^{q_n-1}(\Delta R^n_{q,\varsigma})^2\Big) \le K$. Since $(\Delta R^n_{q,\varsigma})^2 - (\Delta R'^n_{q,\varsigma})^2 = 2\mathfrak{J}^n_{q,\varsigma}\Delta R^n_{q,\varsigma} - (\mathfrak{J}^n_{q,\varsigma})^2$,

we apply the CS inequality across blocks, using $\mathbb{P}(\mathcal{E}^n_{t,\varsigma}) \to 1$ and $\mathbb{P}(N(\epsilon)^n_{t,\varsigma} > 0) \to 0$. Let $n \to \infty$ first for fixed $\epsilon > 0$, and then let $\epsilon \downarrow 0$. This yields the convergence in the lemma. The proof is now complete. □

Lemma SA.15. *Let Assumption (S-HONV) hold. Assume that $d_n, q_n \to \infty$, $L_n\Delta_n \to 0$, and $\overline{k}_n/d_n \to 0$. Then, for $\varsigma \in \{+,-\}$, we have $\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}(\Delta R'^n_{q,\varsigma})^2 \xrightarrow{\mathbb{P}} \frac{2\widetilde{\sigma}^2_{t,\varsigma}}{3\alpha_{t,\varsigma}}$.*

Proof. With w^n_k defined after (SA.22), let $g^n_k := (w^n_k)^2/d^2_n \in [0,1]$ for $1 \le k \le 2d_n - 4\overline{k}_n$, and $g^n_0 = g^n_{2d_n-4\overline{k}_n+1} := 0$. Let $e^n_m := \Delta(n,m) - \frac{\Delta_n}{\alpha^n_{m-1}}$ and $\xi^n_m := e^n_m - \mathbb{E}(e^n_m|\mathcal{I}^n_{m-1})$. For each block, set

$$\mathfrak{K}^n_{q,\varsigma} := \sum_k g^n_k \int_{t^n_{a^n_{q,\varsigma}+k-1}}^{t^n_{a^n_{q,\varsigma}+k}} \widetilde{\sigma}^2_s \mathrm{d}s, \quad \mathfrak{L}^n_{q,\varsigma} := (\Delta R'^n_{q,\varsigma})^2 - \mathfrak{K}^n_{q,\varsigma},$$

where here and below the sums over k run over $1 \le k \le 2d_n - 4\overline{k}_n$. Since $\Delta(n,m) = \Delta_n/\alpha_{t,\varsigma} + \Delta_n((\alpha^n_{m-1})^{-1} - \alpha^{-1}_{t,\varsigma}) + \mathbb{E}(e^n_m|\mathcal{I}^n_{m-1}) + \xi^n_m$, we have $\mathfrak{K}^n_{q,\varsigma} = \sum_{\ell=1}^5 \mathfrak{K}(\ell)^n_{q,\varsigma}$, where

$$\mathfrak{K}(1)^n_{q,\varsigma} := \sum\nolimits_k g^n_k \int_{t^n_{a^n_{q,\varsigma}+k-1}}^{t^n_{a^n_{q,\varsigma}+k}} (\widetilde{\sigma}^2_s - \widetilde{\sigma}^2_{t,\varsigma})\mathrm{d}s, \quad \mathfrak{K}(2)^n_{q,\varsigma} := \widetilde{\sigma}^2_{t,\varsigma}\sum\nolimits_k g^n_k\, \mathbb{E}\Big(e^n_{a^n_{q,\varsigma}+k}|\mathcal{I}^n_{a^n_{q,\varsigma}+k-1}\Big),$$

$$\mathfrak{K}(3)^n_{q,\varsigma} := \widetilde{\sigma}^2_{t,\varsigma}\sum\nolimits_k g^n_k\, \xi^n_{a^n_{q,\varsigma}+k},\ \mathfrak{K}(4)^n_{q,\varsigma} := \frac{\widetilde{\sigma}^2_{t,\varsigma}\Delta_n}{\alpha_{t,\varsigma}}\sum\nolimits_k g^n_k,\ \mathfrak{K}(5)^n_{q,\varsigma} := \widetilde{\sigma}^2_{t,\varsigma}\Delta_n\sum\nolimits_k g^n_k\Big(\frac{1}{\alpha^n_{a^n_{q,\varsigma}+k-1}} - \frac{1}{\alpha_{t,\varsigma}}\Big).$$

Since $\mathfrak{K}(4)^n_{q,\varsigma}$ does not depend on q and $(q_n-1)d_n/L_n \to 1$, and $\sum_k g^n_k = \frac{2}{3}d_n(1+O(\overline{k}_n/d_n))$, we have $\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}\mathfrak{K}(4)^n_{q,\varsigma} \to \frac{2\widetilde{\sigma}^2_{t,\varsigma}}{3\alpha_{t,\varsigma}}$. It remains to show that the other terms are negligible.

In the post-t case, all observation intervals used by the blocks lie in $[t, t+D_{n,+}]$, where $D_{n,+} := t^n_{n_t+L_n+1} - t$. In the pre-$t$ case, on $\mathcal{A}^n_t$, we have $n_t - L_n \ge a_n$, so they lie in $[t-h_n, t)$, where $D_{n,-} := h_n = CL_n\Delta_n$. In both cases, each observation interval is used by at most two blocks, and there are at most $2L_n$ pairs (q,k). Since $\widetilde{\sigma}$ is bounded and $0 \le g^n_k \le 1$, (iii) in Assumption (O) gives the following bound, on $\mathcal{A}^n_t$ in the pre-t case:

$$\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}\sum_{\ell\in\{1,2,5\}}|\mathfrak{K}(\ell)^n_{q,\varsigma}| \le K\Big(1+\frac{D_{n,\varsigma}}{L_n\Delta_n}\Big)\sup_u\big(|\widetilde{\sigma}_u - \widetilde{\sigma}_{t,\varsigma}| + |\alpha^{-1}_u - \alpha^{-1}_{t,\varsigma}|\big) + K\Delta^\eta_n \xrightarrow{\mathbb{P}} 0.$$

Here the supremum is over $[t, t+D_{n,+}]$ for $\varsigma = +$ and $[t-h_n, t)$ for $\varsigma = -$. By Lemma SA.1, $D_{n,+} = O_P(L_n\Delta_n) = o_P(1)$. Thus the convergence follows from the one-sided continuity of $\widetilde{\sigma}$ and $1/\alpha$ at t, the bound $D_{n,-} = CL_n\Delta_n$, and $\mathbb{P}(\mathcal{A}^n_t) \to 1$.

Next, $\mathbb{E}\big((\xi^n_m)^2\big) \le K\Delta^2_n$ by (iii) in Assumption (O). In the post-t case, each ξ^n_m appears in at most two blocks, so martingale-difference orthogonality and the boundedness of $\widetilde{\sigma}$, give $\mathbb{E}\Big((\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}\mathfrak{K}(3)^n_{q,+})^2\Big) \le KL^{-1}_n$. In the pre-$t$ case, we define $\mathfrak{K}(3;s)^n_{q,-}$ by replacing $a^n_{q,-}$ in $\mathfrak{K}(3)^n_{q,-}$ with $a(s)^n_q$; on $\mathcal{A}^n_t$, it coincides at $s = s^*_n$ with $\mathfrak{K}(3)^n_{q,-}$. Then, we have $\mathbb{E}\big((\mathfrak{K}(3;s)^n_{q,-})^2\big) \le Kd_n\Delta^2_n$ and, $\mathbb{E}\big((\mathfrak{K}(3;s+1)^n_{q,-} - \mathfrak{K}(3;s)^n_{q,-})^2\big) \le Kd^{-1}_n\Delta^2_n$ as $|g^n_k - g^n_{k-1}| \le 2/d_n$. Thus (SA.21) with $\kappa = 2$ and $\varpi_n = d_n\Delta^2_n$, and the CS inequality, give

$\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{A}_t^n\}}\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}|\mathfrak{K}(3)_{q,-}^n|\Big) \le Kd_n^{-1/2}$. Thus, in both cases $\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}\mathfrak{K}(3)_{q,\varsigma}^n \xrightarrow{\mathbb{P}} 0$.

Now consider $\mathfrak{L}_{q,\varsigma}^n$ in the post-t case. Expanding the square in $(\Delta R'^n_{q,+})^2$, by optional sampling, the diagonal terms minus $\mathfrak{K}_{q,+}^n$ have zero conditional mean by Itô's isometry, and the cross terms by the martingale property, each conditionally on $\mathcal{I}^n$ at the later left endpoint. Hence $\mathbb{E}\Big(\mathfrak{L}_{q,+}^n|\mathcal{I}_{a_{q,+}^n}^n\Big) = 0$, and $\mathfrak{L}_{q,+}^n$ is $\mathcal{I}_{a_{q+2,+}^n}^n$-measurable. The BDG inequality, the boundedness of $\widetilde{\sigma}$, and (iii) in Assumption (O) give $\mathbb{E}\big((\Delta R'^n_{q,+})^4\big) + \mathbb{E}\big((\mathfrak{K}_{q,+}^n)^2\big) \le Kd_n^2\Delta_n^2$, so $\mathbb{E}\big((\mathfrak{L}_{q,+}^n)^2\big) \le Kd_n^2\Delta_n^2$. As terms two blocks apart are orthogonal,

$$\mathbb{E}\bigg(\Big(\sum\nolimits_{q=1}^{q_n-1}\mathfrak{L}_{q,+}^n\Big)^2\bigg) \le 3\sum\nolimits_{q=1}^{q_n-1}\mathbb{E}\big((\mathfrak{L}_{q,+}^n)^2\big) \le Kq_nd_n^2\Delta_n^2,$$

and hence $\mathbb{E}\Big((\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}\mathfrak{L}_{q,+}^n)^2\Big) \le Kq_n^{-1}$.

In the pre-t case, we define $\mathfrak{K}(s)_{q,-}^n$ and $\mathfrak{L}(s)_{q,-}^n := (\Delta R'(s)_{q,-}^n)^2 - \mathfrak{K}(s)_{q,-}^n$ by replacing $a_{q,-}^n$ with $a(s)_q^n$; on $\mathcal{A}_t^n$, $\mathfrak{L}(s_n^*)_{q,-}^n = \mathfrak{L}_{q,-}^n$. By the stopping-time property after (SA.19), the post-t argument applies to $\mathfrak{L}(s)_{q,-}^n$ uniformly in $s \in \mathcal{S}_n \cup \{0,\dots,d_n-1\}$ and $q \in \mathcal{Q}_n$: it is centered conditionally on $\mathcal{I}_{a(s)_q^n}^n$, it is $\mathcal{I}_{a(s)_{q+2}^n}^n$-measurable, and $\mathbb{E}\big((\mathfrak{L}(s)_{q,-}^n)^2\big) \le Kd_n^2\Delta_n^2$. For the phase increments, $\Delta R'(s+1)_{q,-}^n - \Delta R'(s)_{q,-}^n$ is a Brownian integral whose integrand is bounded by Kd_n^{-1} and supported on at most $2d_n$ observation intervals, so its fourth moment is at most $Kd_n^{-2}\Delta_n^2$; by the CS inequality, $\mathbb{E}\big(((\Delta R'(s+1)_{q,-}^n)^2 - (\Delta R'(s)_{q,-}^n)^2)^2\big) \le K\Delta_n^2$. Moreover, $\sum_k|g_k^n - g_{k-1}^n| \le 2$, and each $\int_{t_{m-1}^n}^{t_m^n}\widetilde{\sigma}_s^2\mathrm{d}s$ has second moment at most $K\Delta_n^2$, so $\mathbb{E}\big((\mathfrak{K}(s+1)_{q,-}^n - \mathfrak{K}(s)_{q,-}^n)^2\big) \le K\Delta_n^2$. Hence $\mathfrak{L}(s+1)_{q,-}^n - \mathfrak{L}(s)_{q,-}^n$ has second moment at most $K\Delta_n^2$, and it has the same centering and measurability properties.

For $0 \le p < d_n$, let $\widetilde{\mathfrak{L}}(p)_{t,-}^n := \frac{1}{L_n\Delta_n}\max_{|m|\le r_n}\big|\sum_{q=1+m}^{q_n-1+m}\mathfrak{L}(p)_{q,-}^n\big|$. Splitting the sums into even and odd q and applying Doob's inequality over $q \in \mathcal{Q}_n$, we obtain $\mathbb{E}\Big((\widetilde{\mathfrak{L}}(p)_{t,-}^n)^2\Big) \le K|\mathcal{Q}_n|d_n^2\Delta_n^2/(L_n\Delta_n)^2 \le Kq_n^{-1}$ and $\mathbb{E}\Big((\widetilde{\mathfrak{L}}(p+1)_{t,-}^n - \widetilde{\mathfrak{L}}(p)_{t,-}^n)^2\Big) \le K|\mathcal{Q}_n|\Delta_n^2/(L_n\Delta_n)^2 \le Kq_n^{-1}d_n^{-2}$. Then (SA.20) with $\kappa = 2$ and $\varpi_n = q_n^{-1}$ gives $\mathbb{E}\Big(\max_{0\le p\le d_n-1}(\widetilde{\mathfrak{L}}(p)_{t,-}^n)^2\Big) \le Kq_n^{-1}$. On $\mathcal{A}_t^n$, the translation property and $|m(s_n^*)| \le r_n$ give $\frac{1}{L_n\Delta_n}|\sum_{q=1}^{q_n-1}\mathfrak{L}_{q,-}^n| \le \widetilde{\mathfrak{L}}(p(s_n^*))_{t,-}^n \le \max_{0\le p\le d_n-1}\widetilde{\mathfrak{L}}(p)_{t,-}^n$. As $\mathbb{P}(\mathcal{A}_t^n) \to 1$, it follows that $\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}\mathfrak{L}_{q,\varsigma}^n \xrightarrow{\mathbb{P}} 0$ in both cases.

Since $(\Delta R'^n_{q,\varsigma})^2 = \mathfrak{K}_{q,\varsigma}^n + \mathfrak{L}_{q,\varsigma}^n$, combining these limits proves the lemma. □

Lemma SA.16. *Let Assumption (S-HONV) hold, and let $\delta_n = d_n^2\Delta_n \to \delta \in [0,\infty]$. If $d_n, q_n \to \infty$, $L_n\Delta_n \to 0$, $k_n \wedge p_n \to \infty$, $(k_n \vee p_n)/d_n \to 0$, and $v > 2$, we have $\frac{\widetilde{\Sigma}_{t,\varsigma}^n}{1+\delta_n} \xrightarrow{\mathbb{P}} \frac{\Gamma_{t,\varsigma}^2\mathfrak{S}}{1+\delta} + \frac{\delta}{1+\delta}\cdot\frac{\widetilde{\sigma}_{t,\varsigma}^2}{3\alpha_{t,\varsigma}}$.*

Proof. Let $\psi_{q,\varsigma}^n := d_n(\Delta U_{q,\varsigma}^n)^2 - d_n\mathbb{E}\big((\Delta U_{q,\varsigma}^n)^2|\mathcal{H}_{q,\varsigma}^n\big)$. By (SA.23), $\mathbb{E}\big((\psi_{q,\varsigma}^n)^2\big) \le K$, and $\psi_{q,\varsigma}^n$ is $\mathcal{H}_{q+2,\varsigma}^n$-measurable, since the indices of $\Delta U_{q,\varsigma}^n$ are smaller than $m(q+1)_{t,\varsigma}^n$, with $\mathbb{E}\big(\psi_{q,\varsigma}^n|\mathcal{H}_{q,\varsigma}^n\big) = 0$. As terms two blocks apart are orthogonal, we get $\mathbb{E}\Big((\frac{1}{q_n}\sum_{q=1}^{q_n-1}\psi_{q,\varsigma}^n)^2\Big) \le Kq_n^{-1}$. By Lemma SA.13, the left-hand side of (SA.25) tends to zero, since $d_n\Delta_n \to 0$ and $\overline{k}_n/d_n \to 0$. The block times $T(q)_{t,\varsigma}^n = t_{m(q)_{t,\varsigma}^n}^n$, $1 \le q < q_n$, lie in $(t, t_{n_t+L_n}^n]$ if $\varsigma = +$, and in $(t-h_n, t)$ on $\mathcal{A}_t^n$ if $\varsigma = -$. Hence Lemma SA.1 and the one-sided continuity of Γ give $q_n^{-1}\sum_{q=1}^{q_n-1}\Gamma_{T(q)_{t,\varsigma}^n}^2 \xrightarrow{\mathbb{P}} \Gamma_{t,\varsigma}^2$. Since $\widetilde{\Sigma}_{t,\varsigma}^{U,n} =$

$\frac{1}{2q_n}\sum_{q=1}^{q_n-1}(\psi^n_{q,\varsigma}+d_n\mathbb{E}\big((\Delta U^n_{q,\varsigma})^2|\mathcal{H}^n_{q,\varsigma}\big))$ and $\mathbb{P}(\mathcal{E}^n_{t,\varsigma})\to 1$, we obtain $\widetilde{\Sigma}^{U,n}_{t,\varsigma}/(1+\delta_n)\xrightarrow{\mathbb{P}}\Gamma^2_{t,\varsigma}\mathfrak{S}/(1+\delta)$.

Next, we consider the cross terms, with all sums over $1\le q<q_n$. By (SA.1), the independence of $\mathcal{F}^{(0)}$ and $\mathcal{G}$, and the triangle inequality over the indices of both blocks, $\mathbb{E}\Big(\mathbb{E}\big(\Delta U^n_{q,\varsigma}|\mathcal{H}^n_{q,\varsigma}\big)^2|\mathcal{F}^{(0)}\Big)\le Kd_n^{-2}(\sum_{r\ge 2\bar{k}_n}(1+r)^{-v})^2\le Kd_n^{-2}$. Also, $\mathbb{E}\big((\Delta U^n_{q,\varsigma})^2|\mathcal{F}^{(0)}\big)\le Kd_n^{-1}$ by (SA.23). We split $\Delta U^n_{q,\varsigma}$ into $\mathbb{E}\big(\Delta U^n_{q,\varsigma}|\mathcal{H}^n_{q,\varsigma}\big)$ and the remainder, apply the CS inequality to the first part and the same orthogonality to the second, and use $\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{E}^n_{t,\varsigma}\}}\sum_q(\Delta R^n_{q,\varsigma})^2\Big)\le KL_n\Delta_n$ from (SA.24) to get for $L_n\le 2q_nd_n$ that

$$\frac{d_n}{q_n}\mathbb{E}\Big(\mathbf{1}_{\{\mathcal{E}^n_{t,\varsigma}\}}\Big|\sum\nolimits_q\Delta U^n_{q,\varsigma}\Delta R^n_{q,\varsigma}\Big|\Big)\le\frac{Kd_n}{q_n}\Big(\frac{q_n^{1/2}}{d_n}\vee\frac{1}{d_n^{1/2}}\Big)(L_n\Delta_n)^{1/2}\le K\big((d_n\Delta_n)^{1/2}\vee(\delta_n/q_n)^{1/2}\big).$$

Since $\big((d_n\Delta_n)^{1/2}\vee(\delta_n/q_n)^{1/2}\big)/(1+\delta_n)\to 0$, and $\mathbb{P}(\mathcal{E}^n_{t,\varsigma})\to 1$, Markov's inequality gives $\widetilde{\Sigma}^{UR,n}_{t,\varsigma}/(1+\delta_n)\xrightarrow{\mathbb{P}}0$.

Then, since $\widetilde{\Sigma}^{R,n}_{t,\varsigma}=\frac{d_nL_n\Delta_n}{2q_n}\cdot\frac{1}{L_n\Delta_n}\sum_{q=1}^{q_n-1}(\Delta R^n_{q,\varsigma})^2$ and $d_nL_n\Delta_n/q_n=\delta_n(1+o(1))$, combining Lemma SA.14 and Lemma SA.15 gives $\widetilde{\Sigma}^{R,n}_{t,\varsigma}/(1+\delta_n)\xrightarrow{\mathbb{P}}\frac{\delta}{1+\delta}\cdot\frac{\widetilde{\sigma}^2_{t,\varsigma}}{3\alpha_{t,\varsigma}}$. Since $\widetilde{\Sigma}^n_{t,\varsigma}=\widetilde{\Sigma}^{U,n}_{t,\varsigma}+\widetilde{\Sigma}^{R,n}_{t,\varsigma}+2\widetilde{\Sigma}^{UR,n}_{t,\varsigma}$, these three limits complete the proof. □

SA.5 Proof of the Consistency of the Asymptotic Variance Estimator

The proofs of Theorems 3.1 to 3.3 and Corollary 3.1 are given in Section SA.3. Here we prove Theorem B.1.

Proof of Theorem B.1. Fix $(t,\varsigma)\in\mathcal{T}\times\{+,-\}$. Under Condition A.5, Lemmas SA.12 and SA.16 apply to each pair $(L_{n,\ell},d_{n,\ell})$, $\ell\in\{1,2\}$, because $d_{n,\ell}\wedge(d_{n,\ell}\Delta_n)^{-1}\asymp\Delta_n^{-\bar{\iota}_\ell}$ with $\kappa\vee\varrho<\bar{\iota}_\ell<v\kappa$, and $L_{n,\ell}\Delta_n\asymp\Delta_n^{1-\iota_\ell-\theta_\ell}\to 0$. These lemmas give $\frac{1}{1+\delta_{n,\ell}}\Sigma(\ell)^n_{t,\varsigma}\xrightarrow{\mathbb{P}}\frac{\Gamma^2_{t,\varsigma}\mathfrak{S}}{1+\delta_\ell}+\frac{\delta_\ell}{1+\delta_\ell}\cdot\frac{\widetilde{\sigma}^2_{t,\varsigma}}{3\alpha_{t,\varsigma}}$. As $\delta_{n,\ell}\to\delta_\ell$, this implies $\epsilon^n_\ell:=\Sigma(\ell)^n_{t,\varsigma}-\Gamma^2_{t,\varsigma}\mathfrak{S}-\frac{\delta_{n,\ell}\widetilde{\sigma}^2_{t,\varsigma}}{3\alpha_{t,\varsigma}}=o_P(1+\delta_{n,\ell})$.

Substituting into the definitions of $\widehat{\mathfrak{S}}(1)^n_{t,\varsigma}$ and $\widehat{\mathfrak{S}}(2)^n_{t,\varsigma}$ gives

$$\widehat{\mathfrak{S}}(1)^n_{t,\varsigma}-\Gamma^2_{t,\varsigma}\mathfrak{S}=\frac{\delta_{n,2}\epsilon^n_1-\delta_{n,1}\epsilon^n_2}{\delta_{n,2}-\delta_{n,1}},\qquad\widehat{\mathfrak{S}}(2)^n_{t,\varsigma}-\frac{\widetilde{\sigma}^2_{t,\varsigma}}{3\alpha_{t,\varsigma}}=\frac{\epsilon^n_1-\epsilon^n_2}{\delta_{n,1}-\delta_{n,2}}.$$

Condition A.5 ensures that $\delta_1,\delta_2\in[0,\infty]$ are distinct and at least one is finite, so $(1+\delta_{n,1})(1+\delta_{n,2})/|\delta_{n,1}-\delta_{n,2}|=O(1)$, and both right sides are $o_P(1)$. Since $c_n\to c$, it follows that $\widehat{\mathfrak{S}}^n_{t,\varsigma}=(\widehat{\mathfrak{S}}(1)^n_{t,\varsigma}+c_n\widehat{\mathfrak{S}}(2)^n_{t,\varsigma})/(1+c_n)\xrightarrow{\mathbb{P}}\mathfrak{S}_{t,\varsigma}$. □

SB Additional Empirical Results

SB.1 Trade Filter

The 2016–2025 estimates in Section 5 of Li and Zhang (2026) use SPY TAQ trades from 2,493 regular sessions, excluding 21 NYSE early-close days. We retain trades with a zero correction indicator,

a finite positive price, a timestamp in $[09{:}30, 16{:}00)$, and none of the sale-condition codes listed in Table SB.1. Odd-lot trades remain eligible. We then apply a median-based variant of the neighborhood screen method proposed in Brownlees and Gallo (2006) to the observed prices p. For the i-th trade in one day, let $\mathcal{M}_i$ be the set of trades j on that day with $|j - i| \leq 100$; it contains 201 trades except near the open and the close, where it is truncated. We define m_i as the median of p_j over $j \in \mathcal{M}_i$, and s_i as the larger of \$0.01 and the median of $|p_j - m_j|$ over $j \in \mathcal{M}_i$, where each m_j is taken over its own window $\mathcal{M}_j$. We remove trade i when $|p_i - m_i| > 3s_i/\Phi^{-1}(0.75) + 0.02$, where Φ is the standard normal distribution function, and keep trades with fewer than 40 observations for either median. Overall, the trade filter removes only 0.699% of trades, with the breakdown in Table SB.1.

Table SB.1: Share of SPY trades removed by the filter, 2016–2025.

Criterion	Number of Removed Trades	Share of Total Trades (%)	Share of Removed Trades (%)
Cancelled, late, or amended	26,655	0.002	0.3
Nonfinite or nonpositive price	0	0.000	0.0
Outside the regular session	2	0.000	0.0
Derivatively priced (4)	5,182,831	0.430	61.5
Contingent trade (7 or V)	777,219	0.064	9.2
Sold out of sequence (Z)	575,903	0.048	6.8
Cash trade (C)	37,461	0.003	0.4
Average-price trade (B)	10,612	0.001	0.1
Late extended-hours sale (U)	961	0.000	0.0
Price far from its neighbors	1,820,369	0.151	21.6
Total removed	8,432,013	0.699	100.0
Retained	1,197,101,562	99.301	

SB.2 Announcement Calendar and Sample Construction

Table SB.2 gives the rule that assigns each of the 2,493 regular sessions of 2016–2025 to the announcement groups of Section 5.1 of Li and Zhang (2026), or to the omitted days. The 21 NYSE early-close sessions are excluded. The 18 further families in Table SB.2 are the University of Michigan consumer sentiment, preliminary and final, Fed chair testimony, the Jackson Hole chair speech, existing and pending home sales, new home sales, JOLTS, factory orders, construction spending, business inventories and wholesale trade, the BLS preliminary payroll benchmark revision, the Conference Board leading economic index, the NAHB housing market index, and the Richmond Fed manufacturing index, all at 10:00; FOMC minutes and the Monthly Treasury Statement at 14:00; and Federal Reserve G.19 consumer credit at 15:00. The core-news and FOMC groups do not exclude releases of these families. Release dates are from the University of Michigan release-date table, federalreserve.gov, the Kansas City Fed, BLS notices, Investing.com event histories, and ALFRED vintage dates.

Table SB.2: Assignment of the regular sessions of 2016–2025 to the announcement groups.

Group	Rule	Days
Core news	ISM Manufacturing, ISM Services, or Conference Board Consumer Confidence release at 10:00, and no scheduled FOMC announcement	344
FOMC	Scheduled FOMC policy announcement at 14:00, and no core 10:00 release	68
Michigan sentiment	Neither core type, and a preliminary or final University of Michigan consumer sentiment release at 10:00	233
Comparison	Neither core type, and no release of any of the 18 further families listed above	900
Omitted	Both a core 10:00 release and a scheduled FOMC announcement	11
Omitted	Neither core type, no Michigan release, and at least one release of the other 16 families	937
Total		2,493

SB.3 Empirical Implementation

Figures 4 and 5 of Li and Zhang (2026) use the differencing horizons $\mathcal{K} := \{3, 4, 5\}$ on five-minute and one-minute grids, respectively. On each side of a grid time, the estimation window contains the $l_n := \min(\bar{l}_n, N_{t\pm})$ trades nearest that time within the adjacent grid interval. Here, we set $\bar{l}_n = 300$, and $N_{t\pm}$ is the total number of trades in the adjacent grid interval on the two sides of grid time t, respectively. We omit any side with fewer than 150 trades. We compute the point estimate directly from (7) of Li and Zhang (2026). We set $k_n = \min(\mathcal{K})$, $p_n = |\mathcal{K}|$, and $\overline{k}_n = \max(\mathcal{K}) + 1$. Section SB.4 reports results for alternative bandwidths and horizons. The same horizon set is used in the point estimate and in all auxiliary variance blocks on that side.

We compute the two-scale variance estimate as described in Appendix B of Li and Zhang (2026). Let N denote the number of filtered trades on the day. On each side of the grid time t, let $N(5)_{t\pm}$ and $N(15)_{t\pm}$ be the number of trades within 5 and 15 minutes of t, respectively. Each count stops at the nearest of three time boundaries: another recorded release, the NYSE/Nasdaq closing deadline in force on that date, or the session boundary. These boundaries are the same on the 1-minute and 5-minute grids. For the small-scale estimator for the asymptotic variance, let $L_1 = \min(\lfloor 5\sqrt{N} \rfloor, N(5)_{t\pm})$, $d_1 = \min(\max(\lfloor L_1/25 \rfloor, 4\overline{k}_n), \lfloor l_n/2 \rfloor)$ where the minimum block length $4\overline{k}_n$ ensures that the trimmed index set in Appendix B is nonempty. For the large scale, when $N(15)_{t\pm} \geq 5l_n$, let $d_2 = \min(\max(\lceil \sqrt{N} \rceil, 2l_n), \lfloor N(15)_{t\pm}/5 \rfloor)$, $L_2 = \min(10d_2, N(15)_{t\pm})$. At each scale $j = 1, 2$, let $q_j = \lfloor L_j/d_j \rfloor$, and use the block origins and inclusive index sets of Appendix B. We compute a variance estimate only when $d_j \geq 4\overline{k}_n$, $N(15)_{t\pm} \geq 5l_n$, and each scale has at least 5

blocks.

We convert the Appendix B estimate to the units of the point estimator: $\widehat{v}_{t\pm} := \widehat{\mathfrak{S}}^n_{t\pm}/u_n^2$, where $u_n^2 = l_n/(1 + l_n^2\Delta_n)$ is evaluated at the actual window length on that side and $\Delta_n = 1/N$. The factor Δ_n cancels in this rescaling. Because the tuning choices ensure $d_1 < l_n \leq d_2$, both auxiliary statistics enter $\widehat{v}_{t\pm}$ with nonnegative weights.

For a significance level $a \in (0, 1)$, the two-sided test rejects $\mathbb{H}_{0,t}$ in (10) of Li and Zhang (2026) when $\frac{|\widehat{\Gamma}^n_{t+} - \widehat{\Gamma}^n_{t-}|}{\sqrt{\widehat{v}_{t+} + \widehat{v}_{t-}}} > z_{1-a/2}$, where $z_{1-a/2}$ is the $(1-a/2)$-quantile of $\mathcal{N}(0, 1)$. We set $a = 0.05$ throughout the empirical analysis. This test is the empirical analogue of the test in Corollary 3.1, with the sum of the two empirical variances in the denominator. We apply it to the eligible observations, those with finite point estimates and finite, positive variance estimates on both sides. At each time, the rejection frequency is the fraction of eligible days on which the test rejects. At a time common to both grids, the variance estimates and the test decisions coincide whenever the spot windows and the horizon sets do. The spot windows can differ, however, when the adjacent minute contains fewer than $\bar{l}_n$ trades. Sections 5.2 and 5.3 of Li and Zhang (2026) state which samples enter the cross-day averages and the rejection frequencies.

The data-dependent block lengths and the cap on the number of large-scale blocks are finite-sample choices that the consistency result in Theorem B.1 does not directly cover. The auxiliary windows do not cross the recorded releases and closing deadlines, but they can still contain unrecorded changes. They also overlap across neighboring grid times, which can make neighboring tests dependent.

For each session, the full-session single-horizon benchmark is the right-side estimator $\widehat{\Gamma}^n_{t+}$ in (7) with $n_t = 0$, $l_n = N$, and $\mathcal{K} = \{5\}$, keeping the trimming by $\overline{k}_n$ and the divisor $l_n = N$. It is an average over transaction indices, not over calendar time. We then average it across the days in each group.

SB.4 Sensitivity to Fixed Horizons and Bandwidth

We assess the sensitivity of announcement-time and midday rejection frequencies to the differencing horizons and the spot-window cap. Table SB.3 crosses $\mathcal{K} \in \{\{3, 4, 5\}, \{6, 7, 8\}, \{3, \ldots, 8\}\}$ with $\bar{l}_n \in \{300, 500\}$; the main specification uses $\mathcal{K} = \{3, 4, 5\}$ and $\bar{l}_n = 300$. The filtering, announcement groups, and two-scale variance construction are the same as before, and the implementation follows Section SB.3 with the indicated horizons and cap. All results use the five-minute grid and the two-sided test at the nominal 5% level. Within each column, the six settings use identical eligible observations, so changes in sample composition do not drive the comparisons.

Table SB.3: Sensitivity of empirical rejection frequencies to fixed horizons and the spot cap.

		Release: nominal 5%			Midday rejection
Horizons	Cap	Core	FOMC	Michigan	5%
$\{3,4,5\}$	300	46.80	45.59	31.76	5.71
$\{3,4,5\}$	500	50.87	58.82	37.34	6.23
$\{6,7,8\}$	300	39.83	42.65	27.90	5.52
$\{6,7,8\}$	500	46.22	61.76	33.05	5.64
$\{3,\ldots,8\}$	300	42.15	47.06	31.33	5.75
$\{3,\ldots,8\}$	500	48.26	58.82	35.19	6.01

All rejection frequencies are percentages at the nominal 5% level. Release-time results use common samples of 344 core-news days at 10:00, 68 FOMC days at 14:00, and 233 Michigan sentiment days at 10:00. Midday results pool 27,740 observations eligible under all six settings at five-minute grid times from 11:00 through 13:30 on the 900 comparison days.

Release-time rejection remains substantial across all six settings, ranging from 39.83% to 50.87% for core news, 42.65% to 61.76% for FOMC announcements, and 27.90% to 37.34% for Michigan sentiment. Increasing the cap raises release-time rejection for every horizon set, with the largest changes for FOMC announcements. Midday rejection varies more modestly, from 5.52% to 6.23%. Thus, the contrast between scheduled releases and midday observations persists, although the magnitudes depend on the tuning parameters. The baseline midday rate here is 5.71%, compared with 5.72% in the main paper, because the common sample contains 27,740 observations rather than all 27,893 observations eligible under the baseline specification. These frequencies describe empirical sensitivity: the calendar-defined comparison days are not known-null observations, so their rejection frequencies do not estimate test size, and release-day frequencies are not power estimates against a known alternative.

SC Additional Simulation Results

We complement the simulation study in Li and Zhang (2026) by varying both the amplitude K_γ and the AR(1) coefficient ρ_χ of the transitory component. All other features of the data-generating process remain unchanged, and each design uses 1,000 replications. Figure SC.1 reports the relative RMSE of $\widehat{\Gamma}_t^n$ at 9:45 for $l_n = 500$. For both noise amplitudes, adding horizons substantially reduces RMSE, especially when serial dependence is strong and the initial horizon k_n is small. With $K_\gamma = 2.5 \times 10^{-4}$, RMSE continues to fall over the reported range in all panels. With the weaker noise $K_\gamma = 10^{-4}$, the gains eventually flatten and some curves turn upward slightly, but the minimum RMSE remains below its single-horizon value in every displayed design. These results confirm the efficiency gains from the multiple-horizon design, but also caution against using too many horizons when the noise signal is weak: variation in the efficient price becomes more influential as the differencing window lengthens and can generate nonnegligible finite-sample error.

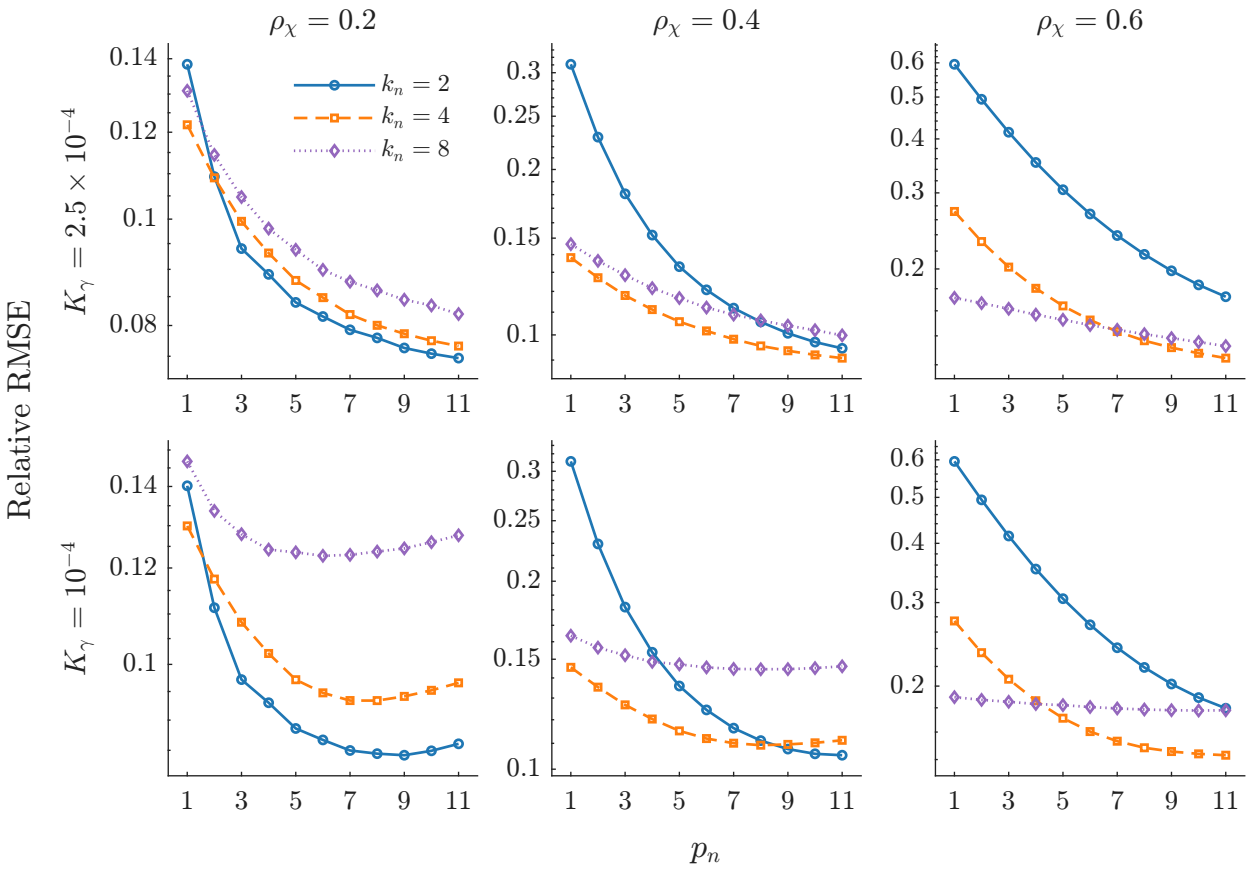


Figure SC.1: Relative RMSE of $\widehat{\Gamma}_t^n$ at $t = 9{:}45$ over 1,000 replications with $l_n = 500$. Rows vary the noise amplitude K_γ, and columns vary the AR(1) coefficient ρ_χ.

Figure SC.2 examines the feasible CLT at 12:00 with $l_n = 500$ under two noise amplitudes and three degrees of serial dependence. We use horizons $\{3, 4, 5\}$ when $\rho_\chi = 0.1$ and $\{6, \ldots, 12\}$ when $\rho_\chi \in \{0.3, 0.5\}$ for both the spot and asymptotic variance estimators. The sample quantiles remain close to the corresponding standard normal quantiles in all six setups. Thus, the normal approximation remains accurate when the noise amplitude is reduced and serial dependence is strengthened.

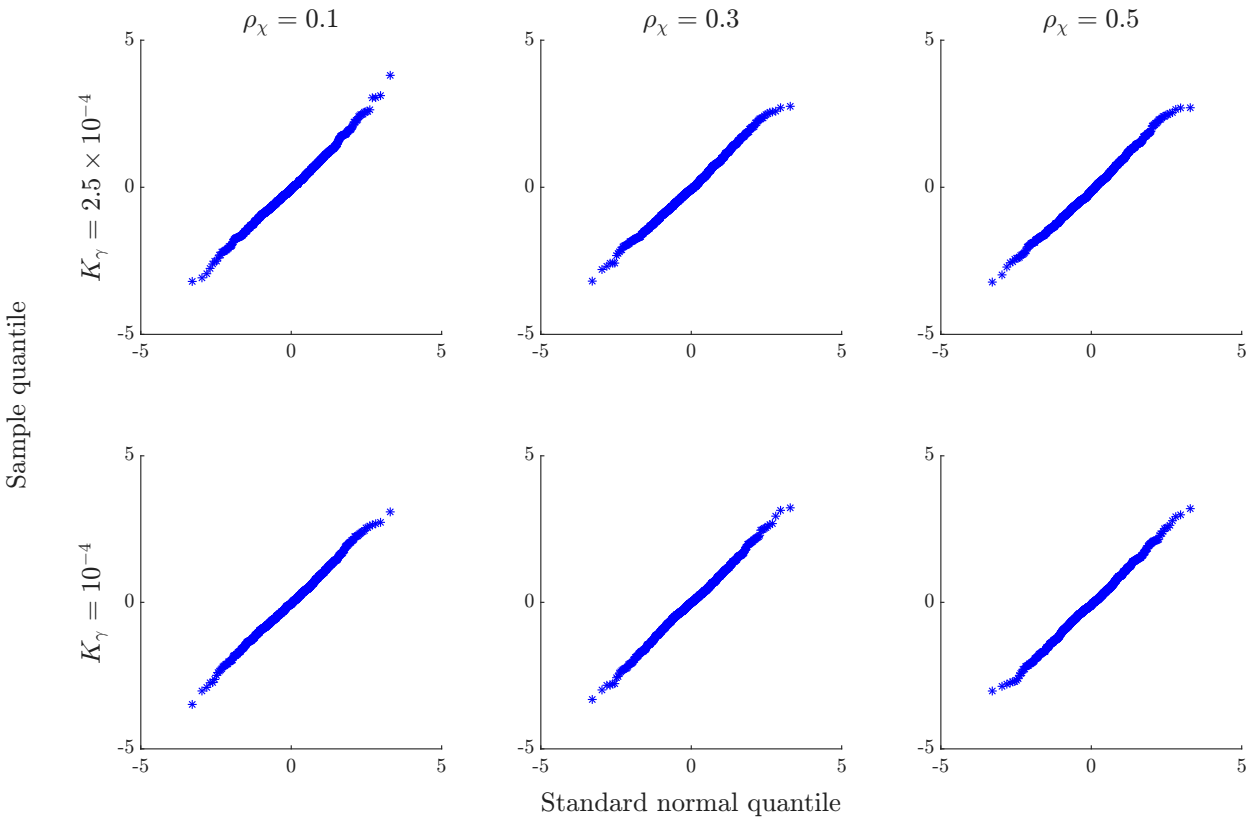


Figure SC.2: QQ plots of the standardized statistic in (9) at $t = 12{:}00$ over 1,000 replications with $l_n = 500$. Rows vary the noise amplitude K_γ, and columns vary the AR(1) coefficient ρ_χ. The differencing horizons are $\{3, 4, 5\}$ for $\rho_\chi = 0.1$ and $\{6, \ldots, 12\}$ for $\rho_\chi \in \{0.3, 0.5\}$.

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